CLASSIFICATION OF POLYNOMIAL SETS

by

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UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "CLASSIFICATION OF POLYNOMIAL SETS" submitted by MOURAD EL HOUSSIENY ISMAIL in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The main purpose of this thesis is to give a certain method of classifying polynomial sets, making use of the linear operator $D_c x^n = c_n x^{n-1}$ where $(c_n, \ n = 0,1,\ldots)$ is a given sequence of numbers. This method of classification contains all other methods that were previously given by Sheffer and others.

As an application some identities were obtained and all orthogonal polynomials which are members of a certain class of polynomial sets were characterized.

A bibliography on Appell polynomials has been included.

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CHAPTER I

INTRODUCTION

The polynomials $p_n(x) = \frac{x^n}{n!}$ $(n = 0,1,\ldots)$ have the property that

$$(1.1) \qquad Dp_n(x) = p_{n-1}(x) \quad (n = 1,2,\ldots),$$

where $D = \frac{d}{dx}$. This led P. Appell [5] in 1880 to consider polynomial sets which satisfy (1.1) (such polynomial sets were called later sets of Appell polynomials). He pointed out several of their interesting properties. For example he proved that a polynomial set $\{p_n(x)\}_0^\infty$ is Appell (i.e. satisfies (1.1)) if and only if

$$\sum_{n=0}^{\infty} p_n(x)t^n = f(t)e^{xt},$$

where $f(t)$ is (at least formally) a power series in t. Well known examples of Appell polynomials are the Hermite and Bernoulli polynomials which may be defined by

$$(1.2) \qquad \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{xt-\frac{1}{2}t^2},$$

and

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t-1}, \tag{1.3}$$

respectively.

Since Appell first introduced them, these polynomial sets have received a great deal of attention (we shall enclose as an appendix a bibliography of Appell polynomials). Various generalizations and analogues of Appell's work appeared. For example polynomial sets which satisfy the functional equation

$$\Delta p_n(x) = p_{n-1}(x), \tag{1.4}$$

where $\Delta f(x) = f(x+1) - f(x)$, have been considered and orthogonal polynomials satisfying (1.4) have also been determined (Carlitz [8]). This is an analogue of a question which occupied several mathematicians who proved that the only Appell polynomial set which is also orthogonal is the set of Hermite polynomials.

Sharma and Chak [19] studied polynomial sets which are the q-analogue of Appell sets, i.e. polynomial sets which satisfy

$$D_q p_n(x) = p_{n-1}(x),$$

where D_q is the q-difference operator defined by

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)\,x} \quad .$$

Ward [26] in his trial "to give a generalization of a large portion of formal parts of algebraic analysis and the calculus of finite differences" replaced the differentiation by a linear and distributive operation which throws x^n into $c_n x^{n-1}$ with $c_o = 0$, $c_1 = 1$ and $c_n \neq 0$ for $n > 1$. If we denote Ward's operator by $D_{c,x}$, then

$$D_{c,x} F(x) = \sum_{n=1}^{\infty} c_n a_n x^{n-1}$$

for any power series

$$F(x) = \sum_{n=o}^{\infty} a_n x^n ,$$

where c stands for the sequence $c_o, c_1, \ldots, c_n, \ldots$.

This idea occurred also to Martin [15] in his work on expansion in terms of a certain class of functions. He considered an operator B defined by

$$(1.5) \qquad B^m F(x) = \frac{1}{2\pi i} \int_{C} \frac{F(z)}{z^{m+1}} \sum_{n=o}^{\infty} \frac{\lambda_o \lambda_1 \cdots \lambda_{n+m}}{\lambda_o \lambda_1 \cdots \lambda_n} \frac{x^n}{z^n} \, dz \qquad (m = 1,2,\ldots),$$

for any single valued analytic function $F(x)$ which is regular at $x = 0$, where c is any circle about the origin within the region of analyticity of $F(z)$, x is any interior point of c and $\lambda_o, \lambda_1, \ldots$ is an infinite sequence of nonzero constants such that

$$\lim_{n \to \infty} |\lambda_o \lambda_1 \ldots \lambda_n|^{\frac{1}{n}} = \infty, \quad \lim_{n \to \infty} |\lambda_n|^{\frac{1}{n}} = 1 .$$

In particular if $F(x) = x^n$ and $m = 1$ then (1.5) reduces to

$$Bx^n = \lambda_n x^{n-1} \quad (n = 1,2,\ldots),$$

$$Bx^o = 0$$

i.e. Martin's operator B reduces to Ward's operator $D_{c,x}$.

Recently Chak [9] considered again polynomial sets satisfying

$$(1.6) \qquad D_{c,x} p_n(x) = p_{n-1}(x),$$

where $D_{c,x}$ is a linear operator such that

$$(1.7) \qquad D_{c,x} x^n = c_n x^{n-1} \quad (n = 0,1,\ldots),$$

with

$$(1.8) \qquad c_o = 0, \quad c_1 = 1 \quad \text{and} \quad c_n \neq 0 \quad (n = 2,3,\ldots) .$$

He calls polynomial sets $\{p_n(x)\}_o^\infty$ satisfying (1.6) "Appell polynomials to the base c ", where c is the sequence $c_o, c_1, \ldots$ of (1.7) and (1.8).

Steffensen [23] and Sheffer [20] gave a further generalization of Appell polynomials by considering polynomial sets $\{p_n(x)\}_o^\infty$ which satisfy

$$(1.9) \qquad J(x,D)p_n(x) = p_{n-1}(x) \quad (n = 1,2,\ldots),$$

where $J(x,D)$ is a linear differential operator of the form

$$(1.10) \qquad J(x,D) = \sum_{r=o}^{\infty} T_r(x)D^{r+1},$$

and $T_r(x)$ is a polynomial in x of degree at most r, for all r. Sheffer [20] classifies polynomial sets according to the maximum degree of the coefficients $T_r(x)$'s in the corresponding linear differential operator $J(x,D)$. He obtained interesting results for the class of polynomials which he called "A-type zero".

Rainville [18] extended Sheffer's work by replacing the

differential operator D in (1.9) and (1.10) by his operator σ, where

$$\sigma = D \prod_{i=1}^{n} (\theta+b_i-1),$$

with

$$\theta = xD,$$

and $b_1, b_2, \ldots, b_n$ are constants.

Ozegov [17] studied polynomial sets $\{p_n(x)\}_o^\infty$ satisfying

$$D^r p_n(x) = p_{n-r}(x) \quad (n = r, r+1, \ldots),$$

where r is a fixed integer. He denotes the class of all such polynomial sets by $A^{(r)}$. Al-Salam and Verma [3] generalized Ozegov's work and considered polynomial sets $\{p_n(x)\}_o^\infty$ satisfying

$$J(D) p_n(x) = p_{n-r}(x) \quad (n = r, r+1, \ldots),$$

where

$$J(D) = \sum_{k=o}^{\infty} a_k D^{k+r}, \quad a_o \neq 0,$$

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and the a_k's are independent of x. They call such polynomials "The generalized Sheffer-Steffensen polynomials" and they denote the class of all such polynomials by $S^{(r)}$.

In this thesis we shall consider the basic ideas in the above-mentioned work by considering operators which include all the work of the above authors.

In Chapter II we review the Sheffer and other known classifications of polynomial sets. In Chapter III we introduce the D_c-type classification which includes all the known classifications as special cases. In Chapter IV we consider the class D_c-type zero polynomials which is the analogue of the class "A-type zero" introduced by Sheffer [20]. In Chapter V we consider the class D_c-type k of polynomials. In Chapter VI we give a characterization of polynomial sets having a Boas and Buck type generating function and we apply that to some polynomials to obtain some identities. In Chapter VII we characterize all orthogonal polynomials generated by

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\mathcal{E}_q(xH(t)),$$

where $\mathcal{E}_q(x)$ is a generalized exponential which will be defined in (7.2).

CHAPTER II

SHEFFER AND OTHER CLASSIFICATIONS

2.1 The Sheffer A-type classification. In what follows we shall denote the differential operator $\frac{d}{dx}$ by D. By a polynomial set $\{p_n(x)\}_o^\infty$ we mean a sequence of polynomials $p_o(x), p_1(x), \ldots$ such that $p_n(x)$ is of degree n, for all n. Now we associate with every polynomial set $\{p_n(x)\}_o^\infty$ a linear differential operator $J(x,D)$ of the form

$$(2.1) \qquad J(x,D) = \sum_{r=o}^{\infty} T_r(x)D^{r+1},$$

such that

$$(2.2) \qquad J(x,D)p_n(x) = p_{n-1}(x) \quad (n = 1,2,\ldots);$$

where $T_r(x)$ is a polynomial of degree at most r, for $r = 0,1,\ldots$.

I.M. Sheffer [20] proved that for any given polynomial set $\{p_n(x)\}_o^\infty$ such an associated linear differential operator $J(x,D)$ exists and is unique. We say with Sheffer that a polynomial set $\{p_n(x)\}_o^\infty$ belongs to the linear differential operator $J(x,D)$, or briefly belongs to J, if and only if J has the form (2.1)

and (2.2) is satisfied. It is true that a polynomial set determines uniquely the associated J-operator but the converse is not true. In fact the next theorem shows that to each J-operator, corresponds infinitely many polynomial sets $\{p_n(x)\}_o^\infty$ such that (2.2) is satisfied.

Theorem 2.1 (Sheffer [20]). *Two polynomial sets* $\{p_n(x)\}_o^\infty$ *and* $\{q_n(x)\}_o^\infty$ *belong to the same J-operator if and only if there exists a sequence of constants* $\{b_n\}_o^\infty$ *such that*

$$b_o \neq 0,$$

and

$$(2.3) \qquad p_n(x) = \sum_{k=o}^{n} b_k q_{n-k}(x) \qquad (n = 0,1,\ldots).$$

Although to each J there are infinitely many polynomial sets $\{p_n(x)\}_o^\infty$ which belong to J, there is a unique polynomial set $\{B_n(x)\}_o^\infty$ which belongs to J and satisfies

$$B_o(0) = 1, \quad B_n(0) = 0 \qquad (n = 1,2,\ldots).$$

Sheffer [20] calls this particular polynomial set $\left\{B_n(x)\right\}_o^\infty$ *the basic polynomial set* associated with the given J-operator.

Sheffer [20] classified polynomial sets in the following way. He says that a polynomial set $\{p_n(x)\}_o^\infty$ is of A-type k $(k < \infty)$ if and only if the maximum degree of the corresponding coefficients $T_r(x)$'s is k. If the degrees of the coefficients $T_r(x)$'s were unbounded, then $\{p_n(x)\}_o^\infty$ is of A-type ∞. We shall follow his definitions.

Several interesting results were obtained for the class A-type zero. In this case $J(x,D)$ will be independent of x, i.e.

$$J(x,D) = J(D) = \sum_{r=o}^{\infty} s_r D^{r+1} \quad (s_o \neq 0). \tag{2.4}$$

Let $H(t)$ be the formal inverse to $J(t)$, i.e. $J(H(t)) \equiv H(J(t)) \equiv t$, then we have

<u>Theorem</u> 2.2 (<u>Sheffer</u> [20]). <u>A polynomial set</u> $\{p_n(x)\}_o^\infty$ <u>is of</u> A-<u>type zero if and only if</u>

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\exp(xH(t)), \tag{2.5}$$

<u>in which</u> $H(t)$ <u>and</u> $A(t)$ <u>have the formal expansions</u>

$$H(t) = \sum_{n=1}^{\infty} h_n t^n \quad \text{with } h_1 \neq 0, \tag{2.6}$$

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(2.7) $$A(t) = \sum_{n=o}^{\infty} a_n t^n \quad \text{with} \quad a_o \neq 0.$$

The series $A(t)$ in the previous theorem is called the determinating series for the given polynomial set $\{p_n(x)\}_o^{\infty}$.

<u>Theorem</u> 2.3 (<u>Sheffer</u> [20]). <u>Let</u> $\{p_n(x)\}_o^{\infty}$ <u>be a polynomial set, then the following conditions are equivalent</u>

(i) $\{p_n(x)\}_o^{\infty}$ <u>is of</u> A-<u>type zero</u>

(ii) <u>There exist sequences of constants</u> $\{\alpha_n\}_o^{\infty}$ <u>and</u> $\{\beta_n\}_o^{\infty}$ <u>such that</u>

(2.8) $$\sum_{k=o}^{n-1} (\alpha_k + x\beta_k) p_{n-k-1}(x) = np_n(x),$$

<u>where the</u> α_k's <u>and</u> β_k's <u>are related to the determinating series</u> $A(t)$ <u>and</u> $H(t)$, <u>the formal inverse to</u> $J(t)$; <u>in the following way</u>

(2.9) $$\sum_{n=o}^{\infty} \alpha_n t^n = A'(t)/A(t)$$

(2.10) $$\sum_{n=o}^{\infty} \beta_n t^n = H'(t) .$$

(iii) <u>There exists a sequence of constants</u> $\{h_n\}_1^{\infty}$ <u>such that</u>

(2.11) $$\sum_{k=1}^{n} h_k p_{n-k}(x) = p_n'(x) \quad (n = 1,2,\ldots),$$

where the constants $h_1, h_2, \ldots$ are those of equation (2.6) in theorem 2.2.

Sheffer also gave the following theorem:

Theorem 2.4 (Sheffer [21] and Thorne [23]). A polynomial set $\{p_n(x)\}_0^\infty$ is of A-type zero if and only if there exists a function $\alpha(x)$ of bounded variation on $(0,\infty)$ such that

(i) The moment integrals

$$\mu_n = \int_0^\infty x^n d\alpha(x) \qquad (n = 0,1,\ldots),$$

all exist.

(ii) $\mu_0 \neq 0$.

(iii) $\{p_n(x)\}_0^\infty$ satisfies

$$(2.12) \qquad \int_0^\infty J^{(r)} p_n(x) d\alpha(x) = \delta_{n,r} \qquad (n,r = 1,2,3,\ldots),$$

where $J = J^{(1)}$ and $J^{(2)}, J^{(3)}, \ldots$ are the iterates of J. Moreover the determinating series $A(t)$ for $\{p_n(x)\}_0^\infty$ is given by

$$(2.13) \qquad A(t) = [\int_0^\infty \{\exp(xH(t)\} d\alpha(x)]^{-1}$$

This is a generalization of an earlier theorem which Thorne [23] gave for Appell polynomials.

Theorem 2.5 (Sheffer [21]). A polynomial set $\{p_n(x)\}_0^\infty$ is of A-type zero if and only if there exists a function $\beta(x)$ of bounded variation on $(0,\infty)$ having the following properties:

(i) The moment integrals

$$b_n = \int_0^\infty x^n d\beta(x),$$

all exist.

(ii) $b_0 \neq 0$.

(iii) For $n = 0,1,2,\ldots$

$$p_n(x) = \int_0^\infty B_n(x+t)d\beta(t), \tag{2.14}$$

where $\{B_n(x)\}_0^\infty$ is the basic set for the corresponding Sheffer operator.

In fact the determinating series $A(t)$ for the polynomial set $\{p_n(x)\}_0^\infty$ of the previous theorem is given by

$$A(t) = \int_0^\infty \exp(xH(t))d\beta(t). \tag{2.15}$$

Sheffer gave characterizations of polynomials of A-type k and A-type zero using the notion of the E-associate of a polynomial set. He calls a sequence of series $\{M_n(x)\}_o^\infty$, <u>the</u> E-<u>associate</u> of a polynomial set $\{p_n(x)\}_o^\infty$ if and only if

$$\exp(tx) = \sum_{n=o}^{\infty} p_n(x)M_n(t). \tag{2.16}$$

It is very easy to see that the E-associate of a given polynomial set $\{p_n(x)\}_o^\infty$ always exists and is unique. Moreover the members of the E-associate have the form

$$M_n(t) = m_{n,n}t^n + m_{n,n+1}t^{n+1} + \dots \quad \text{with} \quad m_{n,n} \neq 0 \tag{2.17}$$

His characterizations are:

<u>Theorem</u> 2.6 (<u>Sheffer</u> [20]). <u>A polynomial set</u> $\{p_n(x)\}_o^\infty$ <u>is of</u> A-<u>type zero if and only if there exist two formal power series</u>:

$$A(t) = \sum_{n=o}^{\infty} a_n t^n \quad \text{with} \quad a_o \neq 0,$$

$$J(t) = \sum_{n=1}^{\infty} c_n t^n \quad \text{with} \quad c_1 \neq 0$$

<u>such that</u>

$$M_n(t) = [J(t)]^n/A(J(t)),$$

where $\{M_n(t)\}_o^\infty$ is the E-associate of $\{p_n(x)\}_o^\infty$.

Theorem 2.7 (Sheffer [20]). *A polynomial set* $\{p_n(x)\}_o^\infty$ *is of A-type* k *if and only if there are* $k+1$ *linear differential operators with constant coefficients* $J_o(D),\ldots,J_k(D)$ *such that*

$$(2.18)\quad \begin{cases} J_m(t) = \sum\limits_{j=m+1}^{\infty} a_{m,j}t^j \quad \text{for} \quad 0 \le m \le k, \\ \\ J_k(t) \not\equiv 0, \end{cases}$$

$$(2.19)\quad a_{o1} + na_{12} + \ldots + n(n-1)\ldots(n-k+1)a_{k,k+1} \neq 0,$$

and

$$(2.20)\quad M_{n+1}(t) = J_o(t)M_n(t) + J_1(t)M_n'(t) + \ldots + J_k(t)M_n^{(k)}(t) \qquad \text{for} \quad n = 0,1,\ldots,$$

where $\{M_n(t)\}_o^\infty$ *is the E-associate of* $\{p_n(x)\}_o^\infty$.

Theorem 2.8 (Sheffer [20]). *A polynomial set* $\{p_n(x)\}_o^\infty$ *is of A-type* k *if and only if the following* $k+1$ *ratios*

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$$(2.21)\quad \Delta_j(t) = \begin{vmatrix} M_n & \cdots & M_n^{(j-1)} & M_{n+1} & M_n^{(j+1)} & \cdots & M_n^{(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{n+k} & \cdots & M_{n+k}^{(j-1)} & M_{n+k+1} & M_{n+k}^{(j+1)} & \cdots & M_{n+k}^{(k)} \end{vmatrix} \Bigg/ \begin{vmatrix} M_n & M_n' & \cdots & M_n^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ M_{n+k} & M_{n+k}' & \cdots & M_{n+k}^{(k)} \end{vmatrix},$$

for $j = 0,1,\ldots,k$,

<u>are independent of</u> n, <u>where</u> $\{M_n(t)\}_o^\infty$ <u>is the</u> E-<u>associate of</u> $\{p_n(x)\}_o^\infty$.

Now we come to two characterizations of A-type k polynomial sets $\{p_n(x)\}_o^\infty$ which are of Brenke type, i.e.

$$(2.22)\qquad \sum_{n=o}^{\infty} p_n(x)t^n = A(t)B(xt),$$

where $A(t)$ and $B(u)$ are formal power series

$$(2.23)\qquad \begin{cases} A(t) = \sum_{n=o}^{\infty} a_n t^n , \\ \\ B(u) = \sum_{n=o}^{\infty} b_n \frac{u^n}{n!} . \end{cases}$$

Boas and Buck [7] proved that (2.22) and (2.23) generate a polynomial set $\{p_n(x)\}_o^\infty$ if and only if

$$a_o b_n \neq 0 \quad \text{for all} \quad n.$$

Huff [13] proved that such a polynomial set is of A-type k if and only if there exists $k+1$ constants $\gamma_o, \gamma_1, \ldots, \gamma_k$ such that

(i) $\gamma_k \neq 0$

and if the g's are defined by

(ii) $b_n g_o g_1 \ldots g_{n-1} = 1$ for $n = 1,2,\ldots,$

then

$$g_o = \gamma_o \,,$$

$$g_n = \sum_{j=o}^{k} \frac{n!}{(n-j)!} \gamma_j \qquad n = 1,2,\ldots,k.$$

Huff and Rainville [14] proved that such a polynomial set $\{p_n(x)\}$ is of A-type k if and only if

$$B(xt) = {}_oF_k(\beta_1,\ldots,\beta_k;\sigma xt).$$

2.2 Sheffer also gave two other methods to classify polynomial sets. Those we explain below.

Let $\{p_n(x)\}_o^\infty$ be a given polynomial set. We associate with it a sequence of polynomials $\{G_n(x)\}_o^\infty$ defined inductively as

$$(2.24)\quad \begin{cases} G_o(x) = p_1'(x)/p_o(x) \\ G_n(x) = [p_{n+1}'(x) - \sum_{k=o}^{n-1} G_k(x)p_{n-k}(x)]/p_o(x) \qquad n = 1,2,\ldots . \end{cases}$$

Clearly the sequence $\{G_n(x)\}_o^\infty$ is unique and

$$(2.25)\qquad p_{n+1}'(x) = \sum_{k=o}^{n} G_k(x)p_{n-k}(x) \quad (n = 0,1,\ldots) .$$

Moreover $G_n(x)$ is a polynomial of degree at most n. If the maximum degree of the $G_n(x)$'s is k $(k < \infty)$ we say that the given polynomial set $\{p_n(x)\}_o^\infty$ is of B-<u>type</u> k. If the degrees of the $G_n(x)$'s were unbounded, we say that $\{p_n(x)\}_o^\infty$ is of B-<u>type</u> ∞.

Sheffer [20] proved that the class A-type zero and B-type zero are identical, but in general there is no close link between the A-type and B-type classifications. As a characterization of finite B-type polynomial sets Sheffer proved:

<u>Theorem</u> 2.8 (<u>Sheffer</u> [20]) <u>A polynomial set</u> $\{p_n(x)\}_o^\infty$ <u>is of</u> B-<u>type</u> k <u>if and only if</u>

$$(2.26)\qquad \sum_{n=o}^{\infty} p_n(x)t^n = A(t)\exp\Big(\sum_{\ell=1}^{k+1} x^\ell H_\ell(t)\Big),$$

<u>where the</u> $H_\ell(t)$'s <u>are of the form</u>

$$H_\ell(x) = \sum_{j=\ell}^{\infty} h_{\ell,j} t^j, \quad h_{\ell\ell} \neq 0.$$

Now we come to the $\mathcal{C}$-type classification. Let $\{p_n(x)\}_0^\infty$ be a polynomial set. We associate with $\{p_n(x)\}_0^\infty$ a sequence of polynomials $\{U_n(x)\}_0^\infty$ such that $U_n(x)$ is of degree at most n and

$$U_1(x) = p_1(x)/p_o(x),$$

$$U_{n+1}(x) = \left[(n+1)p_{n+1}(x) - \sum_{k=1}^{n} U_k(x)p_{n-k}(x)\right]/p_o(x).$$

In other words

$$(2.27) \qquad np_n(x) = \sum_{k=1}^{n} U_k(x)p_{n-k}(x).$$

Clearly for every polynomial set $\{p_n(x)\}_0^\infty$, the associated sequence $\{U_n(x)\}_0^\infty$ is unique. If the maximum degree of the $U_n(x)$'s is $k + 1$, we say that $\{p_n(x)\}_0^\infty$ is of $\mathcal{C}$-type k, otherwise we say it is of $\mathcal{C}$-type ∞. Sheffer [20] proved that for every finite k, the classes B-type k and $\mathcal{C}$-type k are identical. In fact the classes A-type zero, B-type zero and $\mathcal{C}$-type zero are identical.

Huff investigated polynomial sets $\{p_n(x)\}_0^\infty$ which are

Brenke type polynomials. He denotes such polynomial sets by $\{y_n(x)\}_o^\infty$ and we shall follow his notation. He proved

<u>Theorem</u> 2.9 (<u>Huff</u> [13]). <u>A polynomial set</u> $\{y_n(x)\}_o^\infty$ <u>is of</u> B-<u>type</u> k <u>if and only if there are</u> k + 1 <u>constants</u> $\gamma_o, \gamma_1, \ldots, \gamma_k$; <u>with</u> $\gamma_k \neq 0$ <u>such that</u>

$$y_n'(x) = \sum_{\ell=o}^{k} \gamma_\ell x^\ell y_{n-\ell-1}(x) \qquad n = 1,2,\ldots .$$

<u>Moreover a</u> $\{y_n(x)\}_o^\infty$ <u>satisfying</u>

$$y_n'(x) = \sum_{j=o}^{n-1} \gamma_j x^j y_{n-j-1}(x),$$

<u>with</u> $\gamma_j \neq 0$ <u>frequently, is of</u> B-<u>type</u> ∞ .

He also showed that a polynomial set $\{y_n(x)\}_o^\infty$ is of A-type zero (hence of B-type zero and C-type zero) if and only if the corresponding sequence $\{b_n\}_o^\infty$ of equation (2.23) is a geometric progression. Hence he proved that the Hermite polynomials, except for a linear transformation on x, are the only orthogonal polynomials among the A, B and C-type zero sets $\{y_n(x)\}_o^\infty$.

We end this section by stating characterizations of B-type k polynomials.

Theorem 2.10 (Huff [13]). A polynomial set $\{p_n(x)\}_0^\infty$ is a $\{y_n(x)\}_0^\infty$ of finite C-type if and only if

$$np_n(x) = \sum_{j=o}^{k} (\xi_j+\gamma_j x^{j+1})p_{n-j-1}(x) + \sum_{j=k+1}^{n-1} \xi_j p_{n-j-1}(x) \qquad n = 1,2,...$$

where

$$b_{r+1} = \sum_{j=o}^{r} \frac{r!}{(r-j)!} \xi_j b_{r-j},$$

with

$$b_o = 1.$$

Theorem 2.11 (Huff [13]). If a polynomial set $\{p_n(x)\}_0^\infty$ is a $\{y_n(x)\}_0^\infty$ of B-type k (finite or infinite), then

$$np_n(x) = xp_n'(x) + \sum_{j=o}^{n-1} \xi_j p_{n-j-1}(x) \qquad n = 1,2,... \quad ,$$

where the ξ_j's are constants.

2.3 The Rainville σ-classification. Rainville [18] gave an extension of the A-type classification. He defines his σ-operator by

$$\sigma = D \prod_{j=1}^{m} (\theta+b_j-1),$$

where

$$\theta = xD,$$

and $b_1,\ldots,b_m$ are m constants.

In this case both theorem 2.1 and the argument which preceeds it are valid if we replace D by σ. The definition of A-type k, k finite or infinite, will be replaced by σ-type k, k being finite or infinite. Moreover theorem 2.2 will be valid if we replace A-type zero by σ-type zero and $\exp(xH(t))$ by ${}_0F_m(—,b_1,b_2,\ldots,b_m;xH(t))$.

2.4 Ozegov [17] studied polynomial sets $\{p_n(x)\}_0^\infty$ satisfying

$$(2.28) \qquad D^r p_n(x) = p_{n-r}(x) \quad (n = r+1,r+2,\ldots),$$

where r is a fixed integer. He calls such sets "the generalized Appell sets".

Al-Salam and Verma [3] studied polynomial set $\{p_n(x)\}_0^\infty$ satisfying

$$(2.29) \qquad L(D)p_n(x) = p_{n-r}(x) \quad n = r,r+1,\ldots$$

with

(2.30) $$L(D) = \sum_{k=o}^{\infty} a_k D^{k+r} \quad (a_o \neq 0),$$

where r is a fixed integer and the a_k's are constants. They define the class $S^{(r)}$ to be the class of all polynomial sets $\{p_n(x)\}_o^\infty$ which satisfy (2.29) for some $L(D)$ of the form (2.30). The Ozegov class $A^{(r)}$ is the class of polynomial sets $\{p_n(x)\}_o^\infty$ satisfying (2.28). Clearly $A^{(r)} \subset S^{(r)}$. Al-Salam and Verma [3] proved:

<u>Theorem</u> 2.12 (<u>Al-Salam and Verma</u> [3]). <u>A polynomial set $\{p_n(x)\}_o^\infty$ belongs to $S^{(r)}$ if and only if there exist formal power series</u>

$$H(t) = \sum_{j=1}^{\infty} h_j t^j \quad h_1 \neq 0,$$

$$A_s(t) = \sum_{j=o}^{\infty} \alpha_j^{(s)} t^j, \quad 1 \leq s \leq n \quad \text{(not all } \alpha_o^{(s)}\text{'s are zeros)},$$

<u>such that</u>

$$\sum_{j=o}^{r} A_j(t) \exp(xH(\varepsilon_j t)) = \sum_{n=o}^{\infty} p_n(x) t^n,$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ are the r-roots of unity i.e. $\varepsilon_j = e^{2\pi j/r}$ for $j = 1,2,\ldots,r$.

In the special case $L(D) = D^r$ we have $H(t) = t$ and the previous theorem reduces to Ozegov's [17] characterization of the class $A^{(r)}$.

Theorem 2.13 (Al-Salam and Verma [3]). Let $J(D)$ be a linear differential operator of the type (2.4) and let $\alpha(x)$ be a function of bounded variation on $(0,\infty)$ such that $\alpha_o = \int_o^\infty d\alpha(x) \neq 0$. Then a polynomial set $\{p_n(x)\}_o^\infty$ belongs to $S^{(r)}$ if and only if

$$\int_o^\infty \{J(D)\}^m p_n(x) d\alpha(x) = \gamma_{n,m} \qquad m = 0,1,2,\ldots ,$$

where $\gamma_{n,m}$ are entries of an infinite triangular matrix in which

$$\gamma_{n+r,m+r} = \gamma_{n,m},$$

with

$$\gamma_{o,o} = \alpha_o \neq 0.$$

2.5 Orthogonality of certain sets of polynomials. Several mathematicians considered the question of enumerating orthogonal polynomial sets which are at the same time members of a certain

class of polynomial sets. For example Angelesco [4], Carlitz [8], Shohat [22], Toscano [25] and Webster [27] proved that the Hermite polynomial set (1.2) is the only orthogonal polynomial set (up to a linear transformation on x) among the class of Appell polynomials. Later Carlitz [8] proved that the Charlier polynomial sets $\{c_n(x,a)\}_o^\infty$ defined by

$$(2.31) \qquad e^{-t}\left(1+\frac{t}{a}\right)^x = \sum_{n=o}^{\infty} c_n(x,a)t^n$$

is the only orthogonal polynomial set among the class of polynomial sets which satisfy (1.2) up to a linear transformation on x.

Toscano [25] and later Ilieff [28] proved that the only orthogonal polynomial set $\{p_n(x)\}_o^\infty$ which is orthogonal such that the set $\{f_n(x)\}_o^\infty$ defined by

$$f_n(x) = x^n p_n\left(\frac{1}{x}\right),$$

is Appell, is the Laguerre polynomial set.

Meixner [16] and later Sheffer [20] characterized all orthogonal polynomials which are of A-type zero. Their result is

Theorem 2.14 (Meixner [16] and Sheffer [20]). *A polynomial set $\{p_n(x)\}_o^\infty$ of A-type zero is orthogonal if and only if it's*

<u>generating function</u>

$$F(x,t) = \sum_{n=0}^{\infty} p_n(x)t^n,$$

assumes one of the forms

$$F(x,t) = \mu(1-bt)^c \exp\{\frac{d+atx}{1-bt}\} \qquad (abc\mu \neq 0), \tag{2.32}$$

$$F(x,t) = \mu \exp\{t(b+ax)+ct^2\} \qquad (ac\mu \neq 0), \tag{2.33}$$

$$F(x,t) = \mu e^{ct}(1-bt)^{d+ax} \qquad (abc\mu \neq 0), \tag{2.34}$$

$$F(x,t) = \mu(1-\frac{t}{c})^{d_1+\frac{x}{a}}(1-\frac{t}{b})^{d_2-\frac{x}{a}} \qquad (abc\mu \neq 0,\ b \neq c). \tag{2.35}$$

We shall refer to such polynomials as Meixner's polynomials.

Al-Salam [1] proved that all orthogonal polynomials among the finite B-type polynomials are the Meixner polynomials.

Recently, Chihara [10] characterized all orthogonal polynomials which are Brenke type polynomials. We shall not list his results here since we are not going to refer to them in the following discussion.

CHAPTER III

THE D_c-CLASSIFICATION

In this chapter we replace the differential operator D by a different, although related, one which we shall call D_c. Thus we shall attempt to build up a theory of classifying polynomial sets which is analogous to the classification methods mentioned in Chapter II.

For this purpose we shall make use of some ideas that have already been used by M. Ward [26] and others. We say with Ward that a sequence of complex numbers $\{c_n\}_0^\infty$ is a "fundamental sequence" if $c_0 = 0$, $c_n \neq 0$ $(n \geq 1)$. We also use the notation

$$[0]! = 1, \quad [n]! = c_1 c_2 \ldots c_n \quad (n \geq 1).$$

The <u>generalized binomial coefficient</u> is then defined as

$$\left[{n \atop k}\right] = \left[{n \atop n-k}\right] = \frac{[n]!}{[k]![n-k]!}, \qquad (k = 0,1,\ldots,n).$$

These generalized factorials and generalized binomial coefficients are also called factorials and binomial coefficients with respect to the base $c = \{c_n\}_0^\infty$.

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We write further

$$(3.1) \qquad [a+b]^n = \sum_{r=o}^{n} [{n \atop r}] a^r b^{n-r},$$

$$(3.2) \qquad \mathcal{E}(x) = \sum_{n=o}^{\infty} \frac{x^n}{[n]!} .$$

Formulas (3.1) and (3.2) give analogues of the binomial theorem and the exponential function respectively. By $[a+b+c]^n$ we mean $[a+[b+c]]^n$.

If

$$f(x) = \sum_{n=o}^{\infty} f_n x^n,$$

then we define

$$f[a+b] = \sum_{n=o}^{\infty} f_n [a+b]^n,$$

and

$$f(a+b) = \sum_{n=o}^{\infty} f_n (a+b)^n.$$

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A fundamental sequence is called "normal" if and only if

$$[1-1]^n = 0 \quad (n = 1,2,\ldots).$$

Ward [26] proved that

$$[1-1]^{2n+1} = 0 \quad (n = 0,1,\ldots),$$

is valid for any fundamental sequence. Therefore a fundamental sequence is normal if and only if

$$[1-1]^{2n} = 0 \quad (n = 1,2,\ldots),$$

which reduces to

$$(3.3) \qquad [{}^{2n}_{n}] = 2(-1)^{n-1} \sum_{r=o}^{n-1} [{}^{2n}_{r}](-1)^r.$$

He also proved that

$$\mathcal{E}(x)\mathcal{E}(y) = \mathcal{E}(x+y).$$

Definition. To every fundamental sequence $\{c_n\}_o^\infty$ we associate a linear operator $D_{c,x}$ defined by

(3.4) $$D_{c,x}x^n = c_n x^{n-1} \qquad n = 0,1,\dots ,$$

where the result of applying τ to a power series is defined formally by

(3.5) $$D_{c,x}\left(\sum_{n=o}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} a_n c_n x^{n-1}.$$

When no confusion arises we shall drop the subscript x and write D_c for $D_{c,x}$.

<u>Theorem</u> 3.1 <u>Let</u> $\{p_n(x)\}_o^{\infty}$ <u>be a polynomial set. Then there exists a unique operator</u> $\tau(x,D_c)$ <u>of the form</u>

(3.6) $$\tau(x,D_c) = \sum_{r=o}^{\infty} T_r(x) D_c^{r+1},$$

<u>with</u> $T_r(x)$ <u>a polynomial of degree at most</u> r, <u>such that</u>

(3.7) $$\tau(x,D_c)p_n(x) = p_{n-1}(x) \qquad n = 1,2,\dots .$$

<u>Proof</u>. Equation (3.7) shows that $T_r(x)$ is a polynomial in x of degree at most r . In fact we can define the $T_r(x)$'s recursively by

$$\begin{cases} T_o(x)D_c p_1(x) = p_o(x) \\ T_n(x)D_c^{n+1} p_{n+1}(x) = p_n(x) - \sum_{r=o}^{n-1} T_r(x)D_c^{r+1} p_n(x) \quad n = 1,2,\ldots \end{cases}$$

i.e.

$$(3.8) \qquad \begin{cases} T_o(x)D_c p_1(x) = p_o(x) \\ T_n(x) = \dfrac{1}{[n]\ \lambda_{n+1}} \{p_n(x) - \sum_{r=o}^{n-1} T_r(x)D_c^{r+1} p_n(x)\}, \end{cases}$$

where λ_{n+1} is the coefficient of x^{n+1} in $p_{n+1}(x)$, hence $\lambda_{n+1} \neq 0$. It is clear that τ is unique since the $T_n(x)$'s are determined uniquely by (3.8).

In view of the result of this theorem we say that a polynomial set $\{p_n(x)\}_o^\infty$ belongs to the operator $\tau(x,D_c)$, or simply τ, if and only if τ has the form (3.6) and (3.7) is satisfied.

<u>Definition</u>. When the coefficients $T_r(x)$'s of (3.6) have the maximum degree k ($k < \infty$), we shall say that $\{p_n(x)\}_o^\infty$ is of D_c-<u>type</u> k. If the degrees of the $T_r(x)$'s are unbounded we shall say that $\{p_n(x)\}_o^\infty$ is of D_c-type ∞.

We have seen that a polynomial set $\{p_n(x)\}_o^\infty$ belongs to a uniquely determined operator τ. The converse is not true. In

fact we have:

Theorem 3.2 Two polynomial sets $\{p_n(x)\}_o^\infty$ and $\{q_n(x)\}_o^\infty$ belong to the τ-operator if and only if there exists a sequence of constants $\{b_n\}_o^\infty$ with $b_o \neq 0$, such that

$$p_n(x) = \sum_{k=o}^{n} b_k q_{n-k}(x). \tag{3.9}$$

Proof. Let $\{q_n(x)\}_o^\infty$ belong to τ and let $\{p_n(x)\}_o^\infty$ be given by (3.9). Therefore we have

$$\tau\, p_n(x) = \sum_{k=o}^{n} b_k \tau q_{n-k}(x) = \sum_{k=o}^{n-1} b_k q_{n-k-1}(x) = p_{n-1}(x),$$

which shows that $\{p_n(x)\}_o^\infty$ belongs to τ .

Next let both $\{p_n(x)\}_o^\infty$ and $\{q_n(x)\}_o^\infty$ belong to the same τ operators and let

$$p_n(x) = \sum_{k=o}^{n} \eta_{n,k} q_{n-k}(x) \quad (n = 0,1,\ldots) . \tag{3.10}$$

Apply τ to both sides of (3.10) to get

$$p_{n-1}(x) = \sum_{k=o}^{n-1} \eta_{n,k} q_{n-1-k}(x) \quad (n = 1,2,\ldots),$$

i.e.

$$\eta_{n,k} = \eta_{n-1,k} = \cdots = \eta_{o,k} = b_k \quad \text{say,}$$

which shows that the η's are independent of n i.e. depend only on k .

As a consequence of this theorem we see that if there is a polynomial set which belongs to a τ-operator, then there are infinitely many polynomial sets which belong to the same τ-operator. Clearly there is exactly one polynomial set $\{B_n(x)\}_o^\infty$ which belongs to a given τ-operator such that

$$(3.11) \qquad \begin{aligned} B_o(0) &= 1 \\ B_n(0) &= 0 , \end{aligned}$$

such a polynomial set $\{B_n(x)\}_o^\infty$ is called "the basic set of $\tau(x,D_c)$".

Now we give two examples to show that the D_c-classification is different from the A-type classification.

<u>Example</u> 1. Let $c_n = n^2$ then $\{\frac{x^n}{(n!)^2}\}_o^\infty$ is of D_c-type zero but it is of A-type 1, since it belongs to $D + xD^2$.

<u>Example</u> 2. Let

$$c_n = 1 \quad n \geq 1$$

$$c_o = 0.$$

Then $\{\frac{x^n}{2^n}\}_o^\infty$ is of D_c-type 1 and A-type infinity, since it belongs to

$$J(x,D) = 2\sum_{k=1}^{\infty} \frac{(-x)^{k-1}}{k!} D^k,$$

$$\tau(x,D_c) = D_c + xD_c^2$$

where

$$D_c f(x) = \frac{f(x)-f(0)}{x}$$

for any polynomial $f(x)$.

In the rest of this chapter we generalize Al-Salam and Verma's results [3]. We shall study polynomial sets $\{p_n(x)\}_o^\infty$ satisfying

(3.12) $$K(D_c)p_n(x) = p_{n-r}(x) \quad (n = r, r+1, \ldots),$$

for a fixed integer r, where $K(D_c)$ has the form

$$(3.13) \qquad K(D_c) = \sum_{k=o}^{\infty} a_k D_c^{k+r} \quad (a_o \neq 0),$$

where the a_k's are constants. The class of such polynomial sets will be denoted by $G^{(r)}$. The next two theorems are characterizations of the class $G^{(r)}$. It is clear that $G^{(r)}$ reduces to Al-Salam and Verma's class $S^{(r)}$ if $c_n = n$, i.e. $D_c = D$. ~~Moreover for $r = 1$ and $c_n = n$, $G^{(r)}$ reduces to Ozegov's class $A^{(r)}$~~ .

<u>Theorem</u> 3.3 A polynomial set $\{p_n(x)\}_o^{\infty}$ belongs to $G^{(r)}$ if and only if there exist power series

$$(3.14) \qquad H(t) = \sum_{j=1}^{\infty} h_j t^j \quad (h_1 \neq 0),$$

$$(3.15) \quad A_s(t) = \sum_{j=o}^{\infty} \alpha_j^{(s)} t^j \quad (\text{not all } \alpha_o^{(s)} \text{ are zeros}),\ s = 1,2,\ldots,r.$$

such that

$$(3.16) \qquad \sum_{j=1}^{r} A_j(t)\varepsilon(xH(\varepsilon_j t)) = \sum_{n=o}^{\infty} p_n(x)t^n,$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ are the r-roots of unity.

<u>Proof</u>. If (3.14)-(3.16) are satisfied, then clearly (3.12) follows for any $K(D_c)$ of the form (3.13).

Conversely let $\{p_n(x)\}_0^\infty$ belong to $G^{(r)}$ and let

$$K(t) = \{\tau(t)\}^r.$$

Clearly we have

$$\{K(D_c)-t^r\} \sum_{n=0}^{\infty} p_n(x)t^n = 0,$$

and hence

$$(\tau(D_c)-\varepsilon_j t) \sum_{n=0}^{\infty} p_n(x)t^n = 0 \quad \text{for } j = 1,2,\ldots,r,$$

which gives (3.16) where $H(t)$ is the formal inverse to $\tau(t)$.

The second characterization of $G^{(r)}$ is

<u>Theorem</u> 3.4 <u>Let</u> $\alpha(x)$ <u>be a function of bounded variation on</u> $(0,\infty)$ <u>such that</u> $\int_0^\infty d\alpha(x) \neq 0$ <u>and all the moments</u> $\int_0^\infty x^n d\alpha(x)$ <u>exist</u>. <u>Then a polynomial set</u> $\{p_n(x)\}_0^\infty$ <u>belongs to</u> $G^{(r)}$ <u>if and only if</u>

$$(3.17) \qquad \int_0^\infty \{\tau(D_c)\}^m p_n(x)d\alpha(x) = \gamma_{n,m} \quad m = 0,1,\ldots,$$

<u>where the</u> γ_n<u>'s are entries of an infinite triangular matrix in</u>

which

(3.18) $$\gamma_{n+r,m+r} = \gamma_{n,m},$$

and $\tau(D_c)$ is of the form

(3.19) $$\tau(D_c) = \sum_{n=o}^{\infty} a_n D_c^{n+1} \qquad (a_o \neq 0).$$

Proof. Let $\{p_n(x)\}_o^\infty$ belong to $G^{(r)}$ and define $\tau(D_c)$ by

$$\{\tau(t)\}^r = K(t),$$

where

$$K(D_c)p_n(x) = p_{n-r}(x) \qquad n = r, r+1, \ldots \quad .$$

It is clear that $\{(\tau(D_c))^j p_{n+j}(x)\}_{n=o}^\infty$ belongs to $G^{(r)}$ for $j = 0,1,\ldots,$ and

$$(\tau(D_c))^j p_n(x) = (\tau(D_j))^{j+r} p_{n+r}(x) \ .$$

Consequently

$$\gamma_{n,j} = 0 \quad \text{for} \quad j > n,$$

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and

$$\gamma_{n,j} = \gamma_{n+r,j+r}$$

For the converse we shall prove at first that (3.17) and (3.18) define a unique polynomial set. Let

$$\mathcal{E}(xH(t)) = \sum_{n=o}^{\infty} B_n(x)t^n,$$

where

$$H(\tau(t)) = \tau(H(t)) = t.$$

Put

$$\int_o^{\infty} B_n(x)d\alpha(x) = \lambda_n,$$

and assume that (3.17) and (3.18) are satisfied by $\{p_n(x)\}_o^{\infty}$. Write $p_n(x)$ in terms of $B_o(x),\ldots,B_n(x)$ as

$$p_n(x) = \sum_{k=o}^{\infty} p(n,k)B_k(x),$$

where $p(n,k) = 0$ for $k > n$. Therefore

$$(\tau(D_c))^m p_n(x) = \sum_{k=o}^{n-m} p(n,k+m)B_k(x),$$

and hence

$$\sum_{k=o}^{n-m} p(n,k+m)\lambda_k = \gamma_{n,m} \qquad m = 0,1,\ldots,n,$$

which is a system of linear equations in $p(n,k)$ with nonvanishing determinant. Therefore the $p(n,k)$'s are uniquely determined i.e. $\{p_n(x)\}_o^\infty$ is unique.

Now (3.17) implies

$$\int_o^\infty (\tau(D_c))^{m+r} p_{n+r}(x)d\alpha(x) = \gamma_{n+r,m+r} = \gamma_{m,n}$$

Therefore

$$(\tau(D_c))^r p_n(x) = p_{n-r}(x)$$

i.e. $\{p_n(x)\}_o^\infty$ belongs to $G^{(r)}$.

<u>Corollary</u>. <u>Let</u> $\Delta = \Delta(\varepsilon_1,\ldots,\varepsilon_r)$ <u>be the Vandermonde determinant</u>

of $\varepsilon_1,\ldots,\varepsilon_r$. Then

$$A_j(t) = \frac{\sum_{n=0}^{\infty} t^n \left(\sum_{p=0}^{n-1} \Delta_{p+1,p+1}\gamma_{n+p,p} \right)}{\Delta \sum_{s=0}^{\infty} t^s \varepsilon_j^s \lambda_s},$$

where $\Delta_{p,s}$ is the cofactor of the (k,s) entry in Δ and $A_j(t)$, $\gamma_{n,m}$, ε_j, λ_j have the same meaning as before.

Clearly if c_n is propertional to n, the class $G^{(r)}$ reduces to Al-Salam and Verma's class $S^{(r)}$. Now we give examples to show that the classes $S^{(r)}$ and $G^{(r)}$ may intersect without being identical.

Example 1. Let

$$(3.19) \quad \begin{cases} c_{4n} = 4n & (n = 0,1,\ldots), \\ c_{4n+1} = 4n+1 & (n = 0,1,\ldots), \\ c_{4n+2} = 4(4n+2) & (n = 0,1,\ldots), \\ c_{4n+3} = 4(4n+3) & (n = 0,1,\ldots), \end{cases}$$

be a fundamental sequence. It is easy to see that

$$3\mathcal{E}(2xt) + \mathcal{E}(2ixt) + \mathcal{E}(-2ixt) + \mathcal{E}(-2xt) = \frac{13}{4} e^{xt} + (1-\frac{3i}{4})e^{ixt} + (1+\frac{3i}{4})e^{-ixt} + \frac{3}{4}e^{-xt}$$

which shows that $S^{(4)}$ and $G^{(4)}$ have a nonvoid intersection for the fundamental sequence given by (3.19). In fact the polynomial set $\{p_n(x)\}_0^\infty$ defined by

$$
\begin{aligned}
p_{4n}(x) &= \frac{6}{n!}\, x^{4n} && (n = 0,1,\ldots)\\
p_{4n+1}(x) &= \frac{4}{(4n+1)!}\, x^{4n+1} && (n = 0,1,\ldots),\\
p_{4n+2}(x) &= \frac{2}{(4n+2)!}\, x^{4n+2} && (n = 0,1,\ldots)\\
p_{4n+3}(x) &= \frac{1}{(4n+3)!}\, x^{4n+3} && (n = 0,1,\ldots),
\end{aligned}
$$

belongs to both $S^{(4)}$ and $G^{(4)}$ for the fundamental sequence (3.19).

Example 2. Consider the Euler Bernstein polynomials $\{S_n(x)\}_0^\infty$ (Ozegov [17]) which belongs to $A^{(2)}$, since

$$\sum_{n=o}^{\infty} S_n(x)t^n = \left(\frac{1+e^{-t}}{e^t+e^{-t}}\right)e^{xt} + \left(\frac{e^t-1}{e^t+e^{-t}}\right)e^{-xt}.$$

It is easy to see that

$$\sum_{n=o}^{\infty} S_n(x)t^n = \left\{\frac{\frac{1+\lambda}{2}\, e^{-t} + \frac{1-\lambda}{2}\, e^{t} + \lambda}{e^t+e^{-t}}\right\}\mathcal{E}(xt) + \left\{\frac{\frac{1-\lambda}{2}\, e^{-t} + \frac{1+\lambda}{2}\, e^{t} - \lambda}{e^t+e^{-t}}\right\}\mathcal{E}(-xt),$$

where the fundamental sequence $\{c_n\}_o^\infty$ is defined by

$$c_{2n} = \frac{2n}{\lambda} \quad (n = 0,1,\ldots),$$

$$c_{2n+1} = (2n+1)\lambda \quad (n = 0,1,\ldots),$$

and λ is any nonzero constant.

CHAPTER IV

THE CLASS D_c-TYPE ZERO

In this chapter we study the class D_c-type zero in details. All the characterizations given below are analogues of known results for the Sheffer class A-type zero.

Let us recall that a polynomial set $\{p_n(x)\}_0^\infty$ is of D_c-type zero if $\tau(D_c)p_n(x) = p_{n-1}(x)$ $(n = 1,2,\ldots)$, where $\tau(t) = \sum_{k=1}^{\infty} \lambda_k t^k$ $(\lambda_1 \neq 0)$. Let us denote by $H(t)$ the formal inverse of $\tau(t)$, i.e. $\tau(H(t)) = H(\tau(t)) = t$.

We first give an analogue of theorem 2.2.

<u>Theorem</u> 4.1 <u>A polynomial set</u> $\{p_n(x)\}_0^\infty$ <u>is of</u> D_c<u>-type zero if and only if</u>

$$(4.1) \qquad \sum_{n=0}^{\infty} p_n(x)t^n = A(t)\hat{\mathcal{E}}(xH(t)),$$

<u>where</u>

$$(4.2) \qquad A(t) = \sum_{0}^{\infty} a_n t^n, \quad a_0 \neq 0.$$

Proof. Let $\{p_n(x)\}_o^\infty$ be a polynomial set having generating function of the form (4.1) with A(t) satisfying (4.2). Let $\tau(t) = \sum_{k=1}^{\infty} \lambda_k t^k$ be the formal inverse of H(t). Therefore we have

$$\tau(D_c) \sum_{n=o}^{\infty} p_n(x)t^n = A(t) \sum_{n=1}^{\infty} \frac{\tau(x^n)}{[n]!} \{H(t)\}^n$$

$$= A(t)\{ \sum_{n=1}^{\infty} \frac{(H(t))^n}{[n]!} \sum_{k=1}^{n} \lambda_k c_n c_{n-1} \cdots c_{n-k+1} x^{n-k}\}$$

$$= A(t)\{ \sum_{k=1}^{\infty} \lambda_k (H(t))^k\}\{ \sum_{n=o}^{\infty} \frac{(H(t))^n}{[n]!} x^n\} = t \sum_{n=o}^{\infty} p_n(x)t^n,$$

since $\tau(H(t)) = t$.

Therefore $\tau p_n(x) = p_{n-1}(x)$ i.e. $\{p_n(x)\}_o^\infty$ belongs to τ .

To show the converse let $\{p_n(x)\}_o^\infty$ be a polynomial set of D_c-type zero and belongs to $\tau(D_c)$. Define a polynomial set $\{q_n(x)\}_o^\infty$ by

$$\mathcal{E}(xH(t)) = \sum_{n=o}^{\infty} q_n(x)t^n,$$

where H(t) is the formal inverse to $\tau(t)$. It follows from the first part of the proof that $\{q_n(x)\}_o^\infty$ is of D_c-type zero and belongs to τ . Therefore there is a sequence $\{a_n\}_o^\infty$ with $a_o \neq 0$ such that

$$p_n(x) = \sum_{k=o}^{n} a_k q_{n-k}(x).$$

Hence

$$\sum_{n=o}^{\infty} p_n(x)t^n = \sum_{n=o}^{\infty} t^n \sum_{k=o}^{n} a_k q_{n-k}(x)$$

$$= A(t)\left(\sum_{n=o}^{\infty} q_n(x)t^n\right),$$

where

$$A(t) = \sum_{n=o}^{\infty} a_n t^n, \quad a_o \neq 0,$$

i.e.

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\mathcal{E}(xH(t)).$$

It is clear that the formal power series solution of the operational equation

$$D_c f(x) = af(x),$$

where a is a constant, is

$$f(x) = B\mathcal{E}(ax),$$

where B is an arbitrary constant. On the other hand $D_c\mathcal{E}(ax) = a\mathcal{E}(ax)$ for any constant a.

Now we give a characterization of D_c-type zero polynomial sets.

Theorem 4.2 *A polynomial set $\{p_n(x)\}_o^\infty$ is of D_c-type zero if and only if there exists a sequence of constants $\{h_n\}_1^\infty$ such that*

$$(4.3) \qquad D_c p_n(x) = \sum_{k=1}^{n} h_k p_{n-k} \qquad n = 1,2,\ldots,$$

with $h_1 \neq 0$.

Proof. Let $\{p_n(x)\}_o^\infty$ satisfy (4.3) with $h_1 \neq 0$ and let

$$\sum_{n=o}^{\infty} p_n(x)t^n = F(x,t).$$

Therefore we have

$$D_c(F(x,t)) = \sum_{n=o}^{\infty} D_c(p_n(x))t^n = \sum_{n=1}^{\infty}\sum_{k=1}^{n} h_k p_{n-k}(x)t^n$$

$$= \sum_{k=1}^{\infty} h_k t^k \sum_{k=n}^{\infty} p_{n-k}(x)t^{n-k} = H(t)F(x,t),$$

i.e.

$$D_cF(x,t) = H(t)F(x,t),$$

which has the solution

$$F(x,t) = A(t)\mathcal{E}(xH(t)),$$

where

$$A(t) = \sum_{o}^{\infty} a_n t^n,$$

is an arbitrary power series in t.

In this case $A(t)$ will satisfy

$$A(t) = \sum_{n=o}^{\infty} p_n(0)t^n,$$

i.e. $a_o \neq 0$.

Conversely let $\{p_n(x)\}_o^\infty$ be of D_c-type zero, then we have

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\mathcal{E}(xH(t))$$

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by theorem 4.1. Applying D_c to both sides of the previous equation we get

$$\sum_{n=1}^{\infty} D_c p_n(x) t^n = H(t)A(t)\varepsilon(xH(t)) = H(t) \sum_{n=o}^{\infty} p_n(x) t^n,$$

i.e.

$$\sum_{n=1}^{\infty} D_c p_n(x) t^n = (\sum_{k=1}^{\infty} h_k t^k)(\sum_{\ell=o}^{\infty} p_\ell(x) t^\ell) = \sum_{n=o}^{\infty} (\sum_{k=1}^{n} h_k p_{n-k}(x)) t^n.$$

Therefore we get

$$D_c p_n(x) = \sum_{k=1}^{n} h_k p_{n-k}(x),$$

with $h_1 \neq 0$.

Clearly the sequence $\{h_k\}_1^{\infty}$ of theorem 4.2 and the function $H(t)$, the formal inverse to the corresponding $\tau(t)$; are related by

$$H(t) = \sum_{k=1}^{\infty} h_k t^k.$$

The next two theorems are generalizations of theorems 2.4 and 2.5 respectively.

<u>Theorem</u> 4.3 <u>A polynomial set</u> $\{p_n(x)\}_0^\infty$ <u>is of</u> D_c<u>-type zero if and only if there exists a function</u> $\alpha(x)$, <u>of bounded variation on</u> $(0,\infty)$, <u>having the following properties</u>

(i) <u>The moments</u> $\mu_n = \int_0^\infty x^n d\alpha(x)$ <u>all exist,</u>

(ii) $\mu_o \neq 0$,

(iii) $\int_0^\infty \tau^\ell \; p_n(x) \; d\alpha(x) = \delta_{n\ell}$.

<u>Moreover the corresponding determinating series</u> $A(t)$ <u>is related to the moments by</u>

$$(4.4) \quad A(t) = \{\sum_{n=o}^{\infty} \mu_n \frac{(H(t))^n}{[n]!}\}^{-1} = \{\int_0^\infty \mathcal{E}(xH(t))d\alpha(x)\}^{-1}$$

<u>Proof</u>. Let $\{p_n(x)\}_0^\infty$ be a polynomial set of D_c-type zero. Define the μ_n's by (4.4). Clearly $\mu_o = \frac{1}{a_o}$, where

$$A(t) = \sum_{n=o}^{\infty} a_n t^n.$$

It is also clear that the μ_n are well defined. Now by a theorem due to Boas [6] there is a function $\alpha(x)$ of bounded variation on $(0,\infty)$ such that

$$\mu_n = \int_0^\infty x^n d\alpha(x).$$

Let $\{p_n(x)\}_o^\infty$ belong to $\tau(D_c)$, i.e.

$$\tau(D_c)p_n(x) = p_{n-1}(x).$$

Therefore
$$\tau^r p_n(x) = \begin{cases} 0 & \text{if } n < r \\ p_{n-r}(x) & \text{if } n \geq r \end{cases},$$

and (iii) is trivially satisfied for $n < r$. Now we consider the case $n \geq r$. In this case we have

$$\int_o^\infty \tau^r p_n(x)d\alpha(x) = \int_o^\infty p_{n-r}(x)d\alpha(x) = \int_o^\infty p_\ell(x)d\alpha(x)$$

$$= \eta_\ell \quad \text{(say)} \quad \text{for} \quad \ell = 0,1,\ldots .$$

$$\int_o^\infty \left(\sum_o^\infty p_\ell(x)t^\ell\right)d\alpha(x) = \sum_{\ell=o}^\infty \eta_\ell t^\ell,$$

i.e.

$$A(t)\int_o^\infty \mathcal{E}(xH(t))d\alpha(x) = \sum_{\ell=o}^\infty \eta_\ell t^\ell,$$

which gives, by (4.4),

$$\eta_o = 1, \quad \eta_\ell = 0 \quad \ell = 1,2,\ldots,$$

and hence (iii) is valid.

Conversely let $\{p_n(x)\}_o^\infty$ be a polynomial set such that there exists a function $\alpha(x)$ of bounded variation on $(0,\infty)$ such that (i), (ii) and (iii) are satisfied, where $\tau = \tau(D_c)$ is a τ-operator. We shall show that $\{p_n(x)\}_o^\infty$ belongs to τ and hence is of D_c-type zero. Let

$$\{\tau(D_c)\}^\ell = \sum_{k=o}^{\infty} \lambda_{\ell,\ell+k} D_c^{\ell+k},$$

and let

$$p_n(x) = \sum_{r=o}^{n} a_{n,r} x^r.$$

Substituting for $p_n(x)$ and $\{\tau(D_c)\}^\ell$ in (iii) we get

$$(4.5) \qquad \int_o^\infty \sum_{k=o}^{\infty} \lambda_{\ell,\ell+k} \{D_c\}^{\ell+k} \left(\sum_{r=o}^{n} a_{n,r} x^r\right) d\alpha(x) = \delta_{n\ell}.$$

Clearly the case $\ell > n$ is trivial, hence we consider the case $\ell \leq n$. It is evident that

$$\{D_c\}^j x^r = \frac{[r]!}{[r-j]!} x^{r-j} \quad \text{for} \quad r \geq j.$$

The L.H.S of (4.5) $= \int_{o}^{\infty} \sum_{k=o}^{n-\ell} \sum_{r=k+\ell}^{n} a_{n,r}\lambda_{\ell,\ell+k}\{D_c\}^{\ell+k}x^r d\alpha(x)$

$$= \sum_{r=}^{n} a_{n,r} \sum_{k=o}^{r-\ell} \lambda_{\ell,\ell+k} \int_{o}^{\infty} \{D_c\}^{\ell+k}[x^r]d\alpha(x)$$

$$= \sum_{r=\ell}^{n} a_{n,r} \sum_{k=o}^{n-\ell} \lambda_{\ell,\ell+k} \frac{[r]!}{[r-\ell-k]!} \mu_{r-\ell-k}$$

$$= \sum_{r=\ell}^{n} [r]!a_{n,r} \sum_{m=o}^{r-\ell} \lambda_{\ell,r-m} \frac{\mu_m}{[m]!} .$$

Therefore (4.4) reduces to

$$(4.6) \quad \sum_{r=\ell}^{n} [r]!a_{n,r} \sum_{m=o}^{r-\ell} \lambda_{\ell,r-m} \frac{\mu_m}{[m]!} = \delta_{\ell,n} \quad (\ell = 0,1,..,n),$$

which is a system of linear equations whose determinants of coefficients (say) Δ is given by

$$\Delta = \mu_o^{n+1} \prod_{\ell=o}^{n} [\ell]!\lambda_{\ell,\ell}.$$

Since $\mu_o\lambda_{\ell,\ell} \neq 0$, then $\Delta \neq 0$ and hence (4.6) determines the $a_{n,r}$'s uniquely, i.e. $\{p_n(x)\}_o^\infty$ which satisfies the conditions of the theorem exists and is unique. Now comes the final step to show that $\{p_n(x)\}_o^\infty$ belongs to τ .

Clearly we have

$$\int_0^\infty \tau^{\ell+1} p_{n+1}(x)\,d\alpha(x) = \delta_{n+1,\ell+1} = \delta_{n,\ell},$$

and hence

$$\int_0^\infty \tau^{\ell}\tau p_{n+1}(x)\,d\alpha(x) = \delta_{n,\ell},$$

which shows that

$$\tau p_{n+1}(x) = p_n(x).$$

The previous theorem was proved by Chak [9] for the case $H(t) \equiv t$, i.e. for polynomial set $\{p_n(x)\}_0^\infty$ satisfying

$$D_c p_n(x) = p_{n-1}(x).$$

He calls such polynomial sets "Appell polynomial sets to the base $c = \{c_n\}_0^\infty$". They can also be called "generalized Brenke polynomials" or "harmonic sequences to the base $(c) = \{c_n\}_0^\infty$".

<u>Theorem</u> 4.4 <u>A polynomial set</u> $\{p_n(x)\}_0^\infty$ <u>is of</u> D_c<u>-type zero if and only if there exists a function</u> $\beta(x)$ <u>of bounded variation on</u> $(0,\infty)$ <u>such that</u>

(i) <u>The moments</u> $\mu_n = \int_0^\infty x^n d\beta(x)$ <u>all exist with</u> $\mu_0 \neq 0$,

(ii) $p_n(x) = \int_0^\infty B_n[x+t]\,d\beta(t)$,

<u>where</u> $\{B_n(x)\}_0^\infty$ <u>is the corresponding basic set</u>.

<u>Proof</u>. Let (i) be satisfied and let $\{p_n(x)\}_0^\infty$ be defined by (ii). Applying the corresponding $\tau(D_c)$ to both sides of (ii) we get

$$J(D_c)p_n(x) = p_{n-1}(x),$$

since

$$J(D_c)B_n(x) = B_{n-1}(x).$$

Conversely let $\{p_n(x)\}_0^\infty$ be of D_c-type zero and have the generating function $A(t)\mathcal{E}(xH(t))$ where

$$\tau(H(t)) = H(\tau(t)) \equiv t,$$

and

$$A(t) = \sum_0^\infty a_n t^n \quad (a_o \neq 0).$$

Define $\{\mu_n\}_0^\infty$ by

$$\mu_n = [n]!a_n \quad \text{for} \quad n = 0,1,\dots .$$

Thereofe by a theorem due to Boas [6] there exists a function $\beta(x)$ of bounded variation on $(0,\infty)$ whose moments are the μ_m's. Let

$$q_n(x) = \int_0^\infty B_n[x+t]d\beta(t),$$

where

$$\sum_{n=0}^{\infty} B_n(x)t^n = \mathcal{E}(xH(t)).$$

Therefore we have

$$\sum_{n=0}^{\infty} q_n(x)\xi^n = \int_0^\infty \mathcal{E}[xH(\xi)+tH(\xi)]d\beta(t)$$

$$= \mathcal{E}[xH(\xi)] \int_0^\infty \mathcal{E}[tH(\xi)]d\beta(t)$$

where $H(t)$ is considered as a single variable, i.e. $\mathcal{E}[xH(t)] = \mathcal{E}(xH(t))$. Hence

$$\sum_{n=0}^{\infty} q_n(x)\xi^n = \mathcal{E}(xH(t)) \sum_{n=0}^{\infty} \frac{(H(\xi))^n}{[n]!}\mu_n$$

$$= A(t)\mathcal{E}(xH(\xi)),$$

which shows that

$$p_n(x) = q_n(x).$$

Clearly the determinating series $A(t)$ is related to the μ_n's of the previous by

$$A(t) = \sum_{n=o}^{\infty} \frac{\mu_n}{[n]!} t^n.$$

Now we introduce an analogue of the E-associate which we shall call the $\mathcal{E}_c$-associate. Using the notion of $\mathcal{E}_c$-associate we shall prove one more theorem.

Let $\{p_n(x)\}_o^{\infty}$ be a polynomial set and let

$$x^n = \sum_{k=o}^{n} \alpha_k p_k(x) \qquad (\alpha_n \neq 0)$$

$$\therefore \quad \mathcal{E}(tx) = \sum_{n=o}^{\infty} \frac{x^n t^n}{[n]!} = \sum_{n=o}^{\infty} \frac{t^n}{[n]!} \left(\sum_{k=o}^{n} \alpha_k p_k(x)\right) = \sum_{k=o}^{\infty} p_k(x)\left(\alpha_k \sum_{n=k}^{\infty} \frac{t^n}{[n]!}\right),$$

i.e.

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$$\mathcal{E}(tx) = \sum_{n=0}^{\infty} p_n(x)M_n(x)$$

where

$$M_n(x) = \sum_{k=n}^{\infty} m_{n,k}t^k \quad (m_{n,n} \neq 0).$$

The sequence of power series $\{M_n(x)\}_0^\infty$ is called the <u>$\mathcal{E}_c$-associate</u> of $\{p_n(x)\}_0^\infty$.

Now we give a characterization of D_c-type zero polynomials.

<u>Theorem</u> 4.5 <u>A polynomial set</u> $\{p_n(x)\}_0^\infty$ <u>is of</u> D_c<u>-type zero if and only if there exist two formal power series</u>

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \quad (a_0 \neq 0),$$

(4.7)

$$\tau(t) = \sum_{n=1}^{\infty} b_n t^n \quad (b_1 \neq 0),$$

<u>such that</u>

(4.8) $$M_n(t) = (\tau(t))^n/A(\tau(t)),$$

<u>where</u> $\{M_n(t)\}_0^\infty$ <u>is the</u> $\mathcal{E}_c$<u>-associate of</u> $\{p_n(x)\}_0^\infty$.

<u>Proof</u>. Let $\{p_n(x)\}_o^\infty$ be a D_c-type zero polynomial set and suppose that $\{p_n(x)\}_o^\infty$ belongs to $\tau(D_c)$. Let $u = \tau(t)$ and $H(t)$ be the formal inverse to $\tau(t)$. Therefore we have

$$\sum_{n=o}^{\infty} p_n(x)u^n = A(u)\mathcal{E}(xH(u)) = A(u)\mathcal{E}(xt) = \sum_{n=o}^{\infty} p_n(x)\{M_n(t)A(\tau(t))\},$$

and hence (4.7) holds.

On the other hand if (4.7) and (4.8) hold then

$$(4.9) \qquad \mathcal{E}(xt) = \sum_{n=o}^{\infty} p_n(x)M_n(t) = \sum_{n=o}^{\infty} p_n(x) \frac{\{\tau(t)\}^n}{A(\tau(t))} .$$

Therefore (4.9) becomes

$$\sum_{n=o}^{\infty} p_n(x)u^n = A(u)\mathcal{E}(xH(t)),$$

i.e. $\{p_n(x)\}_o^\infty$ is of D_c-type zero.

We now show that the classes A-type zero and D_c-type zero are either disjoint or identical. Clearly they are identical if c_n is propertional to n. To show that let $\{p_n(x)\}_o^\infty$ be a polynomial set of A-type zero and D_c-type zero for a fundamental

sequence $\{c_n\}_o^\infty$, i.e.

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\exp(xH_1(t)),$$

and

$$\sum_{n=o}^{\infty} p_n(x)t^n = B(t)\mathcal{E}(xH_2(t)).$$

Therefore we have

$$A(t)\exp(xH_1(t)) = B(t)\mathcal{E}(xH_2(t)).$$

Substituting $x = 0$ in both sides of the last equation we get

$$A(t) = B(t),$$

and therefore

$$\exp(xH_1(t)) = \mathcal{E}(xH_2(t))$$

which implies

$$c_n = \alpha n \quad n = 0,1,\dots,$$

and

$$H_2(t) = \alpha H_1(t),$$

where α is a constant. Therefore D_c is a constant multiple of the differential operator D and the classes D_c-type zero and A-type zero are identical.

CHAPTER V

THE CLASS D_c-TYPE k

In this chapter we give one characterization of D_c-type k polynomial sets using the notion of the E_c-associate. After giving this characterization we shall study the Brenke type polynomial sets which are also of D_c-type k.

Theorem 5.1 *A polynomial set $\{p_n(x)\}_0^\infty$ is of D_c-type k if and only if there are $k+1$ power series $W_o(x), W_1(x), \ldots, W_k(x)$ such that*

(i) $W_i(x) = \sum_{j=i+1}^{\infty} a_{i,j} x^j$ for $0 \leq i \leq k$,

(ii) $W_k(x) \not\equiv 0$,

(iii) $a_{o1} + c_n a_{12} + \cdots + c_n c_{n-1} \cdots c_{n-k+1} a_{k,k+1} \neq 0$ for $n = 0,1,\ldots,$

and

(iv) $M_{n+1}(x) = \sum_{\ell=o}^{k} W_\ell(x)\{D_c\}^\ell M_n(x)$ for $n = 0,1,\ldots,$

where $\{M_n(t)\}_0^\infty$ is the E_c-associate of $\{p_n(x)\}_0^\infty$.

Proof. Let $\{p_n(x)\}_0^\infty$ be a polynomial of D_c-type k which belongs to $\tau(x,D_c)$. Let

$$\tau(x,t) = \sum_{\ell=o}^{k} x^\ell W_\ell(t), \tag{5.1}$$

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$$[illegible] \quad (4.1)$$

where the $W_\ell(t)$'s are $k + 1$ power series in t whose coefficients are independent of x. Clearly (i) and (ii) are satisfied. Moreover the relation

$$\tau(x,D_c)p_n(x) = p_{n-1}(x)$$

shows that (iii) holds. Therefore we have

$$\tau(x,D_c)\mathcal{E}(xt) = \tau(x,D_c) \sum_{n=o}^{\infty} p_n(x)M_n(t),$$

i.e.

$$(5.2) \qquad \tau(x,D_c)\mathcal{E}(xt) = \sum_{n=o}^{\infty} p_n(x)M_{n+1}(t).$$

On the other hand we have

$$\tau(x,D_c)\mathcal{E}(xt) = \sum_{\ell=o}^{k} x^\ell W_\ell(D_c)\mathcal{E}(xt)$$

i.e.

$$(5.3) \qquad \tau(x,D_c)\mathcal{E}(xt) = \sum_{\ell=o}^{k} W_\ell(t)x^\ell\mathcal{E}(xt).$$

Since

$$x^{\ell}\mathcal{E}(xt) = (D_{c,t})^{\ell}\mathcal{E}(xt) = \sum_{n=o}^{\infty} p_n(x)(D_{c,t})^{\ell}M_n(t),$$

then the required result (iv) follows by equating the right hand sides of (5.2) and (5.3).

Conversely assume that (i),(ii),(iii) and (iv) are satisfied and define $\tau(x,D_c)$ by (5.1). Therefore we have

$$\tau(x,D_c)\mathcal{E}(xt) = \sum_{n=o}^{\infty} \{\tau(x,D_c)p_n(x)\}M_n(t)$$

i.e.

$$\tag{5.4} \tau(x,D_c)\mathcal{E}(xt) = \sum_{n=o}^{\infty} \{\tau(x,D_c)p_{n+1}(x)\}M_{n+1}(t).$$

and

$$\begin{aligned} \tau(x,D_c)\mathcal{E}(xt) &= \sum_{\ell=o}^{k} x^{\ell}W_{\ell}(D_c)\mathcal{E}(xt) \\ &= \sum_{\ell=o}^{k} x^{\ell}W_{\ell}(t)\mathcal{E}(xt) \\ &= \sum_{\ell=o}^{k} W_{\ell}(t)(D_{c,t})^{\ell}\mathcal{E}(xt) \end{aligned}$$

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i.e.

$$(5.5) \qquad \tau(x,D_c)\mathcal{E}(xt) = \sum_{n=o}^{\infty} p_n(x) \sum_{\ell=o}^{k} W_\ell(t)(D_{c,t})^\ell M_n(t)$$

Equating the right hand sides of (5.4) and (5.5) and using (iv) we get

$$\tau(x,D_c)p_n(x) = p_{n-1}(x) \qquad n = 1,2,\ldots$$

i.e. $\{p_n(x)\}_o^\infty$ belongs to $\tau(x,D_c)$ and hence is of D_c-type k.

We now proceed to study D_c-type k polynomials which are of Brenke type, i.e., polynomials sets $\{y_n(x)\}$ which have a generating relation

$$(5.6) \qquad A(t)\phi(xt) = \sum_{n=o}^{\infty} y_n(x)t^n$$

where

$$(5.7) \qquad \begin{cases} \phi(u) = \sum_{n=o}^{\infty} f_n \frac{u^n}{n!} & (f_o \neq 0 \text{ for all } n), \\ A(u) = \sum_{n=o}^{\infty} \alpha_n \frac{u^n}{n!} & (a_o \neq 0). \end{cases}$$

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We note that (5.7) can be also written as

$$(5.7') \qquad \begin{cases} \phi(u) = \sum_{n=o}^{\infty} b_n \dfrac{u^n}{[n]!} \\[2ex] A(u) = \sum_{n=o}^{\infty} a_n \dfrac{u^n}{[n]!} . \end{cases}$$

In fact

$$f_n = b_n \frac{(n!)}{[n]!} , \qquad \alpha_n = a_n \frac{n!}{[n]!}$$

<u>Lemma</u>. <u>A polynomial set</u> $\{y_n(x)\}_o^{\infty}$ <u>belongs to the</u> τ-<u>operator</u>

$$(5.8) \qquad \tau(x,D_c) = \sum_{k=1}^{\infty} \{\ell_{k,o}+x\ell_{k,1}+\ldots+x^{k-1}\ell_{k,k-1}\}D_c^k$$

<u>if and only if</u>

$$(5.9) \qquad \ell_{j,k} = 0 \quad \text{for} \quad k = 0,1,\ldots,j-2.$$

<u>Proof</u>. Let $\{y_n(x)\}_o^{\infty}$ belong to $\tau(x,D_c)$ which is given by (5.8). Let

$$q_n(x) = \frac{x^n}{[n]!} b_n \quad (n = 0,1,\ldots),$$

i.e.

$$\sum_{n=o}^{\infty} q_n(x)t^n = \phi(xt) .$$

Clearly we have

$$p_n(x) = \sum_{m=o}^{n} \frac{a_m}{[m]!} q_{n-m}(x) \qquad n = 0,1,\ldots .$$

Therefore, for $k \leq m$, we have

$$D_c^k p_n(x) = \sum_{m=o}^{n-m} \frac{a_m}{[m]!} b_{n-m} \frac{x^{n-k-m}}{[n-k-m]!} . \tag{5.10}$$

Define $\theta_1, \theta_2, \ldots,$ recursively by

$$b_k \theta_o \theta_1 \theta_2 \cdots \theta_{k-1} = 1 \quad (k = 1,2,\ldots) .$$

Clearly $\theta_k \neq 0$ $(k = 0,1,\ldots)$ since $b_k \neq 0$ $(k = 0,1,\ldots)$.
Define a sequence $\{\ell_{n,n-1}\}_1^{\infty}$ as

$$\theta_n = \sum_{k=1}^{n+1} \frac{[n]!}{[n+1-k]!} \ell_{k,k-1} \qquad n = 0,1,\ldots . \tag{5.11}$$

Now by (5.10) we have

$$\sum_{k=1}^{\infty} \ell_{k,k-1} x^{k-1} \{D_c\}^k [p_n(x)] = \sum_{k=1}^{n} \ell_{k,k-1} x^{k-1} \sum_{m=o}^{n-k} \frac{a_m}{[m]!} b_{n-m} \frac{x^{n-m-k}}{[n-k-m]!}$$

i.e.

$$(5.12) \quad \sum_{k=1}^{\infty} \ell_{k,k-1} x^{k-1} D_c^k p_n(x) = \sum_{m=o}^{n-1} a_m \frac{x^{n-m-1}}{[m]!} b_{n-m} \sum_{k=1}^{n-m} \frac{\ell_{k,k-1}}{[n-m-k]!} .$$

Since $\theta_k = \dfrac{b_k}{b_{k+1}}$, we get for $n \geq m + 1$

$$\sum_{k=1}^{n-m} \frac{\ell_{k,k-1}}{[n-m-k]!} = \frac{\Theta_{n-m-1}}{[n-m-1]!} = \frac{b_{n-m-1}}{b_{n-m}[n-m-1]!} ,$$

substituting in (5.12) we get

$$\sum_{k=1}^{n} \ell_{k,k-1} x^{k-1} D_c^k p_n(x) = p_{n-1}(x),$$

which shows that (5.9) is valid.

Conversely let $\{p_n(x)\}_o^{\infty}$ belong to (5.8) such that (5.9) holds, i.e.

$$(5.13) \quad \sum_{k=1}^{\infty} \ell_{k,k-1} x^{k-1} D_c^k p_n(x) = p_{n-1}(x) \quad n = 1, \ldots ,$$

Since the coefficient of x^{n-1} in the left hand side of (5.12) must be nonzero, then

$$c_n \ell_{10} + c_n c_{n-1} \ell_{21} + \ldots + c_1 c_2 \ldots c_n \ell_{n,n-1} \neq 0 ,$$

i.e. if we define θ_k's by (5.11) then $\theta_k \neq 0$ for all k. We now can define a sequence $b_o, b_1, \ldots, b_n, \ldots$ by

$$b_o = 1, \quad b_{k+1} = b_k/\theta_k \quad (k = 0,1,2,\ldots)$$

Let

$$q_n(x) = \frac{x^n}{[x]!} b_n \qquad n = 0,1,\ldots .$$

Clearly $\{q_n(x)\}$ is of Brenke type since

$$\sum_{n=o}^{\infty} q_n(x) t^n = \sum_{n=0}^{\infty} \frac{b_n}{[n]!} (xt)^n = \phi(xt) , \quad \text{(say)}.$$

Moreover

$$\sum_{k=1}^{n} \ell_{k,k-1} x^{k-1} D_c^k q_n(x) = \sum_{k=1}^{n} \ell_{k,k-1} x^{k-1} b_n \frac{x^{n-k}}{[n-k]!}$$

$$= x^{n-1} \frac{b_n}{[n-1]!} \theta_{n-1}$$

$$= q_{n-1}(x),$$

i.e., $\{p_n(x)\}_o^\infty$ and $\{q_n(x)\}_o^\infty$ belong to the same τ and hence

$$\sum_o^\infty p_n(x)t^n = A(t)\phi(xt).$$

As a direct consequence of this lemma we get

Theorem 5.2 *A polynomial set* $\{y_n(x)\}_o^\infty$ *is of* D_c*-type* k *if and only if there exist* $k + 1$ *constants* $\gamma_o, \gamma_1, \ldots, \gamma_k$, *with* $\gamma_k \neq 0$, *such that if the* θ's *are defined by*

(5.14) $$b_n \theta_o \theta_1 \cdots \theta_{n-1} = 1 \quad n = 1,2,\ldots$$

then $\theta_o = \gamma_o$, *and*

(5.15) $$\theta_n = \sum_{r=o}^{n} \frac{[n]!}{[n-r]!} \gamma_r \quad (n = 1,2,\ldots,k).$$

Now we introduce the following notation

$$[a]_o = 1, \quad [a]_n = a(a+c_1) \ldots (a+c_{n-1}) \quad (n = 1,2,\ldots)$$

As we have mentioned in Chapter 2, Huff and Rainville [14] showed that a Brenke type polynomial set is of A-type k if and only if $\phi(xt)$ of equation (5.4) is

$$\phi(xt) = \sum_{n=0}^{\infty} \frac{(\sigma xt)^n}{n!(\beta_1)_n \cdots (\beta_k)_n},$$

where $\sigma \neq 0, \beta_1, \ldots, \beta_k$ are constants.

We shall study now the D_c analogue of the previous result if we replace $n!$, $(\beta_j)_n$ by $[n]!$ and $[\beta_j]_n$ respectively.

<u>Theorem 5.3</u> <u>If the fundamental sequence</u> $\{c_n\}_0^{\infty}$ <u>is normal then a polynomial set</u> $\{p_n(x)\}_0^{\infty}$ <u>satisfying</u>

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t) \sum_{n=0}^{\infty} \frac{(\sigma xt)^n}{[n]![\beta_1]_n \cdots [\beta_k]_n}$$

<u>is of</u> D_c<u>-type</u> k.

<u>Proof</u>. Let

$$\theta_s = \frac{b_s}{b_{s+1}} = \left(\frac{1}{\sigma}\right) \prod_{j=1}^{k} (\beta_j + c_s) \qquad s = 0,1,\ldots .$$

Define γ's recursively by

$$\gamma_o = \theta_o$$

$$\theta_s = \sum_{j=o}^{s} \frac{[s]!}{[s-j]!} \gamma_j \qquad s = 1,2,\ldots,k.$$

We claim that

$$(5.16) \qquad \gamma_i = \sum_{s=o}^{i} \frac{(-1)^{i-s}}{[s]![i-s]!} \theta_s \qquad i = 0,1,\ldots,k \ .$$

The right hand side of (5.16) is given by

$$\sum_{s=o}^{i} \frac{(-1)^{i-s}}{[s]![i-s]!} \theta_s = \sum_{s=o}^{i} \frac{(-1)^{i-s}}{[s]![i-s]!} \left(\sum_{j=o}^{s} \frac{[s]!}{[s-j]!} \gamma_j \right)$$

$$= \sum_{j=o}^{i} \frac{\gamma_j}{[i-j]!} (-1)^{i-j} \sum_{\ell=o}^{i-j} \left[{i-j \atop \ell}\right] (-1)^{\ell}.$$

But

$$\sum_{\ell=o}^{i-j} \left[{i-j \atop \ell}\right] (-1)^{\ell} = 0,$$

since $\{c_n\}_o^{\infty}$ is a normal sequence. Therefore the right hand side of (5.16) equals γ_i.

Let

$$q_n(x) = \frac{\sigma^n x^n}{[n]![\beta_1]_n \cdots [\beta_k]_n} .$$

Clearly $\{q_n(x)\}_o^\infty$ and $\{p_n(x)\}_o^\infty$ belong to the same τ-operator. Now we claim that $\{q_n(x)\}_o^\infty$ belongs to

$$\tau(x,D_c) = \sum_{j=o}^{k} \gamma_j x^j D_c^{j+1} .$$

For $n \geq 1$ we have

$$\sum_{j=o}^{k} \gamma_j x^j D_c^{j+1} q_n(x) = \sum_{j=o}^{k} \frac{\gamma_j \, {}^n x^{n-1}}{[\beta_1]_n \cdots [\beta_k]_n [n-j-1]!}$$

$$= q_{n-1}(x) \frac{\sigma}{\prod_{j=1}^{k} (\beta_j + c_{n-1})} \sum_{j=o}^{k} \frac{\gamma_j [n-1]!}{[n-j-1]!}$$

$$= q_{n-1}(x) .$$

Therefore $\{q_n(x)\}_o^\infty$, and hence $\{p_n(x)\}_o^\infty$, is of D_c-type k.

Now we give an example to show that the previous theorem is false if the fundamental sequence $\{c_n\}_o^\infty$ is not normal.

<u>Example</u>. Let $\beta_1 = \beta_2 = 1$, $c_n = n^2$ and $p_o(x) = 1$

$$p_n(x) = \frac{x^n}{(n!)^2\{(1+1^2)(1+2^2)\cdots(1+(n-1)^2\}^2} \qquad n = 1,2,\ldots .$$

Clearly

$$\sum_{0}^{\infty} p_n(x)t^n = \sum_{n=o}^{\infty} \frac{x^n t^n}{[n]!\{[1]_n\}^n} .$$

Let $\{p_n(x)\}_o^\infty$ belong to $\tau(x,D_c)$, i.e.

$$\sum_{k=o}^{\infty} T_k(x)D_c^{k+1}p_n(x) = p_{n-1}(x).$$

Therefore

$$(T_1(x)D_c+T_2(x)D_c^2+\ldots)\ \frac{x^n}{n^2} = \{1+(n-1)^2\}^2 x^{n-1} \qquad n = 1,2,\ldots ,$$

which gives

$$T_o(x) = 1, \quad T_1(x) = 3x,$$

$$T_2(x) = 3x^2, \quad T_3(x) = -x^3 .$$

Therefore $\{p_n(x)\}_o^\infty$ is of D_c-type k where $k \geq 3$.

<u>Theorem</u> 5.4 <u>If</u> $c_n \neq c_m$ <u>for</u> $m \neq n$ <u>and</u> $\{y_n(x)\}_o^\infty$ <u>is of</u> D_c<u>-type</u> k, <u>then there exist</u> k <u>parameters</u> $\beta_1,\ldots,\beta_k$ <u>and a nonzero constant</u> σ <u>such that</u>

$$\frac{1}{b_n} = [\beta_1]_n[\beta_2]_n\cdots[\beta_k]_n(\frac{1}{\sigma})^n \qquad n = 0,1,\ldots$$

<u>Proof</u>. Define $\xi_1,\xi_2,\ldots,\xi_k$ by

$$\gamma_k \sum_{j=0}^{k} \xi_j c_s^{k-j} = \theta_s \qquad s = 1,2,\ldots,k, \quad (\text{with } \xi_o = 1), \tag{5.17}$$

where the γ's and θ's are those of Theorem 5.2. The ξ's are well defined since the determinant of the system (5.17) is nonvanishing. Let $\beta_1,\beta_2,\ldots,\beta_k$ be the k roots of

$$z^k - \xi_1 z^{k-1} + \ldots + (-1)^k \xi_k = 0$$

i.e.

$$\gamma_k \prod_{j=1}^{k} (\beta_j + c_s) = \theta_s \qquad s = 1,2,\ldots,k.$$

Therefore

$$b_n = (\frac{1}{\gamma_k})^n \Big/ \prod_{j=1}^{k} [\beta_j]_n,$$

i.e. $\sigma = \gamma_k$ and the result follows.

We end this chapter by giving an example to show that the condition $c_n \neq c_m$ for $m \neq n$ can not be omitted. To see that let

$$c_n = 1 \quad n \neq 0 \quad \text{and} \quad c_o = 0,$$

$$p_o(x) = 1, \qquad p_1(x) = x$$

$$p_2(x) = \frac{x^2}{2}, \; p_n(x) = \frac{9}{2}\left(\frac{x}{3}\right)^n \quad n = 3,4,\ldots$$

Clearly $\{p_n(x)\}_o^\infty$ satisfies

$$\{D_c + xD_c^2 + x^2D_c^3\}p_n(x) = p_{n-1}(x),$$

i.e. $\{p_n(x)\}_o^\infty$ is of D_c-type 2 where

$$D_c x^n = x^{n-1}.$$

Now let us assume the contrary, i.e.

$$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\sum_{n=o}^{\infty}\frac{(\sigma xt)^n}{[\beta_1]_n[\beta_2]_n}.$$

In other words

$$1 + xt + \frac{x^2t^2}{2} + \frac{x^3t^3}{6} + \ldots = A(t)\{1 + \frac{\sigma xt}{\beta_1\beta_2} + \frac{\sigma^2x^2t^2}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)} +$$

$$+ \frac{\sigma^3 x^3 t^3}{\beta_1(\beta_1+1)^2\beta_2(\beta_2+1)^2} + \ldots\} .$$

Therefore

(i) $A(t) = 1,$

(ii) $\dfrac{\sigma}{\beta_1\beta_2} = 1,$

(iii) $\dfrac{\sigma^2}{\beta_1(\beta_1+1)\beta_2(\beta_2+1)} = \dfrac{1}{2},$

(iv) $\dfrac{\sigma^3}{\beta_1(\beta_1+1)^2\beta_2(\beta_2+1)^2} = \dfrac{1}{6}.$

....

From (ii) and (iii) we get

$$\frac{\sigma}{(\beta_1+1)(\beta_2+1)} = \frac{1}{2},$$

while from (iii) and (iv) we get

$$\frac{\sigma}{(\beta_1+1)(\beta_2+1)} = \frac{1}{6},$$

which is a contradiction.

CHAPTER VI

MISCELLANEOUS RESULTS

In this chapter we shall study the consequences of the identities (3.6) and (4.3) for several polynomial sets. At first we give a characterization of polynomial sets having generating functions of the form studied by Boas and Buck [7]. They studied polynomial sets $\{p_n(x)\}_o^\infty$ generated by $A(t)f(xH(t))$, i.e.

$$(6.1) \qquad \sum_{n=o}^{\infty} p_n(x)t^n = A(t)f(xH(t)) ,$$

where

$$(6.2) \qquad A(t) = \sum_{n=o}^{\infty} a_n t^n ,$$

$$(6.3) \qquad H(t) = \sum_{n=1}^{\infty} h_n t^n ,$$

$$(6.4) \qquad f(u) = \sum_{n=o}^{\infty} f_n u^n .$$

They proved that the polynomials $p_o(x), p_1(x), \ldots$ defined by (6.1)

[illegible]

[illegible]

[illegible]

$$[illegible] \qquad [illegible]$$

[illegible]

$$[illegible] \qquad [illegible]$$

$$[illegible] \qquad (2.4)$$

$$[illegible] \qquad (2.5)$$

[illegible]

form a polynomial set if and only if

(6.5) $$a_o h_1 f_n \neq 0 \quad (n = 0,1,2,\ldots).$$

There is no loss of generality to assume $f_o = 1$. We shall associate with every polynomial set $\{p_n(x)\}_o^\infty$ satisfying (6.1)-(6.5) a fundamental sequence $\{c_n\}_o^\infty$ defined as follows

$$c_o = 0, \quad c_{n+1} = \frac{f_n}{f_{n+1}} \quad (n = 0,1,\ldots).$$

Let D_c be the D_c operator associated with the fundamental sequence $\{c_n\}_o^\infty$. Clearly

$$f(xH(t)) = \mathcal{E}(xH(t)),$$

i.e. $\{p_n(x)\}_o^\infty$ is of D_c-type zero and belongs to $\tau(D_c)$ where $\tau(t)$ is the formal inverse to $H(t)$. Therefore all characterization of D_c-type zero polynomials are characterizations of polynomial sets having Boas and Buck type generating functions.

I. <u>Laguerre polynomials</u>. The Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_o^\infty$ have the generating function (Rainville [18], p. 203)

$$\sum_{n=o}^\infty \frac{(c)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x)t^n = \frac{1}{(1-t)^c} {}_1F_1(c;1+\alpha;\frac{-xt}{1-t}).$$

We shall consider the case $c > 1$. Let

$$c_n = \frac{(\alpha+n)n}{c+n-1} \quad (n = 1,2,\ldots),$$

$$p_n(x) = (c)_n \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n}$$

Clearly $\{p_n(x)\}_o^\infty$ is of D_c-type zero with

$$D_c x^n = \frac{(\alpha+n)n}{c+n-1} x^{n-1} .$$

In this case

$$H(t) = \frac{-t}{1-t} ,$$

i.e. $h_n = -1$ for $n = 1,2,\ldots$.

For any polynomial $f(x)$ we have

$$D_c f(x) = f'(x) + \left(\frac{1+\alpha-c}{x}\right) f(x) + \frac{(1-c)(1+\alpha-c)}{x^c} \int_o^x t^{c-2} f(t)dt .$$

Therefore (4.3) gives

$$\frac{(c)_n}{(1+\alpha)_n}\left\{\frac{d}{dx} L_n^{(\alpha)}(x) + \frac{1+\alpha-c}{x} L_n^{(\alpha)}(x) + \frac{(1-c)(1+\alpha-c)}{x^c}\int_o^x t^{c-2} L_n^{(\alpha)}(t)dt\right\}$$

$$= - \sum_{k=o}^{n-1} L_k^{(\alpha)}(x) \frac{\binom{\alpha+n}{n-k}}{\binom{c+n-1}{c-k}} ,$$

which reduces to

$$(6.6) \quad \int_o^x t^{\beta-1} L_n^{(\alpha)}(t)dt = \frac{x^\beta}{\beta} L_n^{(\alpha)}(\beta) + \frac{x^{\beta+1}}{\beta(\alpha-\beta)} \frac{d}{dx} L_n^{(\alpha)}(x)$$

$$+ \frac{x^{\beta+1}}{\beta(\alpha-\beta)} \sum_{k=o}^{n-1} L_k^{(\alpha)}(x) \frac{\binom{n+\alpha}{n-k}}{\binom{n+\beta}{n-k}}$$

where $c = \beta + 1$ i.e. $\beta > 0$.

Since

$$\frac{d}{dx} L_n^{(\alpha)}(x) = - \sum_{k=o}^{n-1} L^{(\alpha)}(x),$$

(Rainvelle [18], p. 202) then (6.6) reduces to

$$(6.7) \quad \int_o^x t^{\beta-1} L_n^{(\alpha)}(t)dt = \frac{x^\beta}{\beta} L_n^{(\alpha)}(x) + \frac{x^{\beta+1}}{\beta(\alpha-\beta)} \sum_{k=o}^{n-1} \left\{\frac{\binom{n+\alpha}{n-k}}{\binom{n+\beta}{n-k}} -1\right\} L_k^{(\alpha)}(x) .$$

II Legendre polynomials. The Legendre polynomials $\{p_n(x)\}_o^\infty$ have the generating function

$$\sum_{n=o}^{\infty} p_n(x)t^n = (1-2xt+t^2)^{-\frac{1}{2}}$$

$$= (1+t^2)^{-\frac{1}{2}} \sum_{n=o}^{\infty} \frac{(\frac{1}{2})_n}{n!} \left(\frac{2\,xt}{1+t^2}\right)^n .$$

The associated D_c-operator is

$$D_c x^n = \frac{n}{n-\frac{1}{2}}\, x^{n-1} ,$$

i.e.

$$D_c f(x) = \frac{f(x)-f(0)}{x} + \frac{1}{2\sqrt{x}} \int_o^x t^{-3/2}[f(t)-f(0)]dt,$$

for any polynomial $f(x)$.

In this case

$$H(t) = \frac{2t}{1+t^2} = 2 \sum_{n=o}^{\infty} (-1)^n t^{2n+1},$$

i.e.

$$h_{2n} = 0$$

$$h_{2n+1} = 2(-1)^n .$$

Therefore (4.3) gives

$$\frac{p_n(x)-p_n(0)}{x} + \frac{1}{2\sqrt{x}} \int_0^x t^{-3/2}[p_n(t)-p_n(0)]dt = 2 \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^{k-1} p_{n+1-2k}(x) ,$$

and hence

$$(6.8) \quad \int_0^x t^{-3/2}[p_n(t)-p_n(0)] = -\frac{2}{\sqrt{x}} \{p_n(x)-p_n(0)\} + 4\sqrt{x} \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^{k-1} p_{n+1-2k}(x) .$$

In particular we have

$$\int_0^1 t^{-3/2}\{p_n(t)-p_n(0)\}dt = \begin{cases} 2(\frac{(\frac{1}{2})_n}{n!} -1) & n = 4\ell \quad \ell = 0,1,\ldots \\ 2 & n = 4\ell+1, \ \ell = 0,1,\ldots \\ 2(\frac{(\frac{1}{2})_{n/2}}{n!} -1) & n = 4\ell+2, \ \ell = 0,1,\ldots \\ -2 & n = 4\ell+3, \ \ell = 0,1,\ldots \end{cases}$$

III <u>Gegenbauer polynomials</u>. The Gegenbauer polynomials $\{c_n^{(\nu)}(x)\}_0^\infty$ have the generating function (Rainville [18], p. 277)

$$\sum_{n=o}^{\infty} c_n^{(\nu)}(x)t^n = (1-2xt+t^2)^{-\nu} .$$

We shall consider only the case $\nu > 0$. The case $\nu = 1$ corresponds to the Tchebicheff polynomials of the second kind which will be considered as a separate case. Thus we shall assume $\nu \neq 1$. Clearly $\{c_n^{(\nu)}(x)\}_o^{\infty}$ is of D_c-type zero with respect to the fundamental sequence $\{\frac{n}{\nu+n-1}\}_o^{\infty}$, since

$$\sum_{n=o}^{\infty} c_n^{(\nu)}(x)t^n = (\frac{1}{1+t^2})^{\nu}\{\sum_{n=o}^{\infty} \frac{(\nu)_n}{n!} (\frac{2t}{1+t^2})^n x^n\} .$$

Therefore the corresponding $H(t)$ and $\tau(D_c)$ are given by

$$H(t) = \frac{2t}{1+t^2}$$

$$\tau(D_c) = \sum_{n=1}^{\infty} (-1)^{n-1}(\frac{1}{2})_n D_c^{2n-1} .$$

In fact

$$D_c f(x) = \frac{f(x)-f(0)}{x} + \frac{1-\nu}{x^{\nu}} \int_o^x t^{\nu-2}[f(t)-f(0)]dt ,$$

for any polynomial $f(x)$. Therefore (4.3) gives

$$(6.9)\quad \frac{c_n^{(\nu)}(x)-c_n^{(\nu)}(0)}{x} + \frac{1-\nu}{x^\nu}\int_o^x t^{\nu-2}[c_n^{(\nu)}(t)-c_n^{(\nu)}(0)]dt$$

$$= 2\sum_{k=1}^{[\frac{n+1}{2}]} (-1)^{k-1} c_{n+1-2k}^{(\nu)}(x) ,$$

i.e.

$$(6.10)\quad \int_o^x t^{\nu-1}\left(\frac{c_n^{(\nu)}(t)-c_n^{(\nu)}(0)}{t}\right)dt = \frac{x^\nu}{\nu-1}\,\frac{c_n^{(\nu)}(x)-c_n^{(\nu)}(0)}{x} + 2\sum_{k=1}^{[\frac{n+1}{2}]}(-1)^k c_{n+1-2k}^{(\nu)}(x)$$

Clearly the case $\nu = \frac{1}{2}$ reduces to the case of Legendre polynomials.

IV <u>The Tchebicheff polynomials</u>. The Tchebicheff polynomials of the second kind $\{T_n(x)\}_o^\infty$ are Gegenbauer polynomials with $\nu = 1$. The previous argument will be valid till we arrive to (6.9) with $\nu = 1$, i.e.

$$T_n(x) = T_n(0) + 2x\sum_{k=o}^{[\frac{n-1}{2}]}(-1)^k T_{n-k-1}(x) .$$

In this case we have

$$T_n(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2} & n \text{ even} \end{cases} ,$$

and hence

$$(6.11)\qquad \begin{cases} T_{2n+1}(x) = 2x \sum\limits_{k=o}^{n} (-1)^k T_{2n-k}(x) \\ \\ T_{2n}(x) = (-1)^n + 2x \sum\limits_{k=o}^{n-1} (-1)^k T_{2n-k-1}(x) \end{cases},$$

V <u>Humbert polynomials</u>. The Humbert polynomials $\{h_n(x)\}_o^\infty$ have the generating function (Rainville [18], p. 146),

$$(1-3xt+t^3)^{-\nu} = \sum_{n=o}^{\infty} h_n(x)t^n .$$

Clearly

$$(1-3xt+t^3)^{-\nu} = (1+t^3)^{-\nu} \sum_{n=o}^{\infty} \frac{(\nu)_n}{n!} \left(\frac{3t}{1+t^3}\right)^n x^n,$$

therefore the associated D_c operator is

$$D_c x^n = \frac{n}{n+\nu-1} x^{n-1} \qquad n = 0,1,\dots .$$

In this case

$$H(t) = \frac{3t}{1+t^3} = 3 \sum_{k=o}^{\infty} (-1)^k t^{3k+1},$$

and (4.3) gives

$$(6.12)\quad \frac{h_n(x)-h_n(0)}{x} + \frac{1-\nu}{x^\nu}\int_0^x t^{\nu-2}\{h_n(t)-h_n(0)\}dt = 3\sum_{k=1}^{[\frac{n+2}{3}]} h_{n+2-3k}(x) \quad (\nu > 0),$$

since for $\nu > 0$

$$D_c f(x) = \frac{f(x)-f(0)}{x} + \frac{1-\nu}{x^\nu}\int_0^x t^{\nu-1}\{\frac{f(t)-f(0)}{t}\}dt,$$

for any polynomial $f(x)$.

VI <u>Bateman's polynomials</u> $z_n(x)$. The Bateman's polynomials $\{z_n(x)\}_0^\infty$ have the generating function (Rainville [18], p. 285),

$$\sum_{n=0}^{\infty} z_n(x)t^n = \frac{1}{1-t}\ {}_1F_1(\tfrac{1}{2};1;\frac{-4xt}{(1-t)^2}).$$

Therefore $\{z_n(x)\}_0^\infty$ is of D_c type zero where

$$D_c x^n = \frac{n^2}{n-\frac{1}{2}}\, x^{n-1} \quad n = 0,1,\ldots$$

In fact

$$D_c f(x) = Df(x) + \frac{f(x)-f(0)}{2x} + \frac{1}{4\sqrt{x}} \int_0^x t^{-\frac{1}{2}}\left(\frac{f(t)-f(0)}{t}\right)dt.$$

In this case

$$H(t) = \frac{-\,4t}{(1-t)^2} = -4 \sum_{k=1}^{\infty} kt^k,$$

and (4.3) gives

$$Dz_n(x) + \frac{z_n(x)-z_n(0)}{2x} + \frac{1}{2\sqrt{x}} \int_0^x t^{-\frac{1}{2}} \frac{z_n(t)-z_n(0)}{2t}\,dt$$

$$= -4 \sum_{k=1}^{n} kz_{n-k}(x) \ .$$

CHAPTER VII

ORTHOGONALITY OF CERTAIN POLYNOMIAL SETS

In this chapter we characterize orthogonal polynomial sets $\{p_n(x)\}_0^\infty$ which are generated by

$$(7.1) \qquad \sum_0^\infty p_n(x)t^n = A(t)\ \mathcal{E}_q(xH(t)),$$

where

$$(7.2) \qquad \mathcal{E}_q(w) = 1 + \sum_{n=1}^\infty \frac{w^n(1-q)^n}{(1-q)(1-q^2)\dots(1-q^n)},$$

$$(7.3) \qquad A(t) = \sum_{n=0}^\infty a_n t^n \quad a_0 \neq 0 \ ,$$

$$(7.4) \qquad H(t) = \sum_{n=1}^\infty h_n t^n \quad h_1 \neq 0.$$

In otherwords we characterize orthogonal polynomials $\{p_n(x)\}_0^\infty$ which are of D_c-type zero with

$$c_n = \frac{q^n-1}{q-1}.$$

We shall denote D_c by D_q in this case. Clearly

$$D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x} ,$$

for any polynomial $f(x)$. Meixner's polynomials will be included as a special case since as q tends to 1 the function $\mathcal{E}_q(w)$ tends to the exponential function $e(w)$.

There is no loss of generality to assume that $a_o = h_1 = 1$. Let α_n be the coefficient of x^n in $p_n(x)$. It follows easily from (7.1) that

$$\alpha_o = 1, \quad \alpha_n = \frac{(1-q)^n}{(1-q)(1-q^2)\dots(1-q^n)} \qquad n = 1,2,\dots .$$

It is also well known that

(7.5) $$D_q(f(x)g(x)) = f(qx)D_q f(x) + g(x)D_q f(x) .$$

By (4.3) we have

(7.6) $$D_q p_n(x) = \sum_{k=1}^{n} h_k p_{n-k}(x).$$

Moreover a polynomial set $\{p_n(x)\}_o^\infty$ is orthogonal if and only if

there are B_n and C_n such that

$$(7.7)\qquad \frac{1}{\alpha_{n+1}} p_{n+1}(x) = (x-B_n)\frac{p_n(x)}{\alpha_n} - \frac{C_n}{\alpha_{n-1}} p_{n-1}(x) \quad (n = 0,1,\ldots;\ p_{-1} = 0),$$

and

$$C_n \neq 0 .$$

In fact $C_n > 0$ if we require orthogonality in the classical sense. Now substituting for α_n in (7.7) we get

$$(7.7')\qquad \left(\frac{1-q^{n+1}}{1-q}\right) p_{n+1}(x) = (x-B_n)p_n(x) - C_n\left(\frac{1-q}{1-q^n}\right) p_{n-1}(x).$$

Applying D_q to both sides of (7.7') we get

$$\left(\frac{1-q^{n+1}}{1-q}\right) \sum_{k=1}^{n+1} h_k p_{n+1-k}(x) = qxD_q p_n(x) + p_n(x) - B_n \sum_{k=1}^{n} h_k p_{n-k}(x)$$

$$+ \left(\frac{1-q}{1-q^n}\right) C_n \sum_{k=1}^{n-1} h_k p_{n-1-k}(x) .$$

Simplifying we get

$$(7.8)\quad \left(\frac{1-q^{n+1}}{1-q}\right)\sum_{k=1}^{n+1} h_k p_{n+1-k}(x) = q\sum_{k=1}^{n} h_k\{B_{n-k}p_{n-k}(x)+\left(\frac{1-q^{n-k+1}}{1-q}\right)p_{n-k+1}$$

$$+ C_{n-k}\left(\frac{1-q}{1-q^{n-k}}\right)p_{n-k-1}(x)] + p_n(x) - B_n\sum_{k=1}^{n} h_k p_{n-k}(x)$$

$$- \left(\frac{1-q}{1-q^{n}}\right)C_n\sum_{k=1}^{n-1} h_k p_{n-k-1}(x)\ .$$

The coefficients of $p_n(x)$ in both sides of (7.8) are identical. Equating coefficients of $p_{n-1}(x)$ in both sides we get

$$(7.9)\qquad B_n - qB_{n-1} = -h_2(1+q^n),$$

whose solution is

$$(7.10)\qquad B_n = bq^n - h_2\left(\frac{1-q^n}{1-q}\right) - nh_2q^n\ .$$

Equating coefficients of $p_{n-2}(x)$ in both sides of (7.8) we get

$$(7.11)\quad \left(\frac{1-q}{1-q^n}\right)C_n - q\left(\frac{1-q}{1-q^{n-1}}\right)C_{n-1} = h_2^2 - h_3 + \{h_2b(1-q)+3h_2^2-(1+q)h_3\}q^{n-1}$$

$$- nh_2^2(1-q)q^{n-1}\ ,$$

whose solution is

$$(7.12)\quad \left(\frac{1-q}{1-q^n}\right)C_n = Cq^n + (h_2^2-h_3)\left(\frac{1-q^n}{1-q}\right) + n\{h_2 b(1-q)+3h_2^2-(1+q)h_3\}q^{n-1}$$

$$- \frac{n(n+1)}{2}\, q^{n-1}(1-q)h_2^2 \ .$$

Clearly as $q \to 1$ we get

$$B_n = b - 2nh_2$$

$$C_n = n\{C+(4h_2^2-3h_3)n\}$$

which corresponds to Meixner's polynomials.

Now we go back to (7.8) and equate coefficients of $p_{n-k-1}(x)$ in both sides to get

$$(7.13)\quad h_{k+2}\left[\left(\frac{1-q^{n+1}}{1-q}\right)-q\left(\frac{1-q^{n-k-1}}{1-q}\right)\right] + h_{k+1}[B_n-qB_{n-k-1}]$$

$$+ h_k\left[\left(\frac{1-q}{1-q^n}\right)C_n-q\left(\frac{1-q}{1-q^{n-k}}\right)C_{n-k}\right] = 0 \ .$$

If $q \to 1$ then (7.13) becomes

$$(7.14)\quad (k+2)h_{k+2} - 2h_2(k+1)h_{k+1} + (4h_2^2-3h_3)kh_k = 0$$

which is identical with equation (4.7), Sheffer [20] page 612 with

his $\alpha = 1$, since $h_1 = 1$, and his $q_{k+1,1}$ replaced by $(k+1)h_{k+1}$. Therefore in this case we have the four cases of Meixner's polynomials.

Now we proceed to study the case $q \neq 1$ and q does not approach 1. Writing $(n+1)$ for n in (7.13) subtracting q times (7.13) from the resulting equation and using (7.9)-(7.12) we get

$$(7.15)\quad (1-q)h_{k+2} - h_2h_{k+1}(1+q^{n+1}-q-q^{n-k}) + h_k[(h_2^2-h_3)(1-q) + \{h_2b(1-q)+3h_2^2 -(1+q)h_3\}(q^n-q^{n-k+1}) - (n+1)(1-q)h_2^2(q^n-q^{n-k+1}) - k(1-q)h_2^2q^{n-k+1}] = 0 .$$

Writing $n+1$ for n in (7.15) and subtracting (7.15) from the resulting equation we get

$$-h_2h_{k+1}[q^{n+1}(q-1)-q^{n-k}(q-1)] + h_k[\{h_2b(1-q)+3h_2^2-(1+q)h_3\}\{q^n(q-1)-q^{n-k+1}(q-1)\} - (n+1)(1-q)h_2^2\{q^n(q-1)-q^{n-k+1}(q-1)\} + k(1-q)^2h_2^2q^{n-k+1} - (1-q)h_2^2(q^{n+1}-q^{n-k+1})] = 0 .$$

Now we consider the case $q \neq 0$. The case $q = 0$ will be considered separately. Dividing by $(1-q)q^n$ we get

$$(7.16)\quad h_2h_{k+1}\{q-\frac{1}{q^k}\} + h_k[\{h_2b(1-q)+3h_2^2-(1+q)h_3\}\{\frac{q}{q^k}-1\}+k(1-q)h_2^2\frac{q}{q^k}$$

$$-qh_2^2\{1-\frac{q}{q^k}\}+(n+1)h_2^2\{1-\frac{q}{q^k}\}] = 0,$$

therefore

$$h_kh_2^2 = 0 \quad \text{for} \quad k = 2,3,\ldots .$$

If $h_2 = 0$, then (7.16) gives $h_3 = 0$. Substituting $k = 3,4,\ldots$ in (7.15) we get $h_4 = h_5 = \ldots = 0$.

Clearly $h_k = 0$ for $k = 2,3,\ldots$ gives the same result.

Therefore if $q \neq 1$ then $H(t) \equiv t$ and the B_n's and C_n's will be given by

$$(7.10')\qquad B_n = bq^n \quad n = 0,1,\ldots$$

$$(7.12')\qquad (\frac{1-q}{1-q^n})C_n = Cq^n \quad n = 0,1,2,\ldots \quad (C \neq 0)$$

Now we proceed to determine the determinating series $A(t)$ for the case $q(1-q) \neq 0$. In this case (7.7') becomes

$$(\frac{1-q^{n+1}}{1-q})p_n(x) = (x-bq^n)p_n(x) - Cq^np_{n-1}(x).$$

Multiplying both sides by t^{n+1} and adding for $n = 0,1,\ldots,$ we get

$$\frac{1}{1-q}\{A(t)\,\mathcal{E}_q(xt)-A(qt)\,\mathcal{E}_q(qxt)\} = xtA(t)\,\mathcal{E}_q(xt) - btA(qt)\,\mathcal{E}_q(qxt)$$

$$- Cqt^2A(qt)\,\mathcal{E}_q(qxt).$$

Using the identity

$$\mathcal{E}_q(qw) = \{1-(1-q)w\}\mathcal{E}_q(w),$$

we get

$$(7.17)\qquad A(t) = A(qt)\{1-b(1-q)t-Cq(1-q)t^2\}.$$

Let

$$1 - b(1-q)z - Cq(1-q)z^2 = (1-\alpha z)(1-\beta z),$$

then (7.17) takes the form

$$A(t) = A(qt)\{(1-\alpha t)(1-\beta t)\}$$

whose solution is

$$A(t) = \{\mathcal{E}_q(\tfrac{t}{1-q})\,\mathcal{E}_q(\tfrac{t}{1-q})\}^{-1}.$$

Therefore $\{p_n(x)\}_0^\infty$ is essentially the set $\{U_n^{(a)}(x)\}$ studied by Al-Salam and Carlitz [2].

Summarizing, we have

<u>Theorem</u> 7.1 <u>The only orthogonal polynomials generated by</u> (7.1) <u>with</u>

$$q(1-q) \neq 0 ,$$

<u>are the polynomials</u> $\{U_n^{(\alpha)}(x)\}_0^\infty$ <u>up to a constant multiple of</u> x.

Now we consider the case $q = 0$ i.e. $c_n = 1$. In this case (7.10), (7.12) and (7.15) become

$$B_n = -h_2 , \tag{7.18}$$

$$C_n = h_2^2 - h_3 , \tag{7.19}$$

and

$$h_{k+2} - h_2 h_{k+1} + (h_2^2 - h_3)h_k = 0 \quad (k = 1,2,\ldots), \tag{7.20}$$

respectively.

For orthogonality in the classical sense we must have,

$$h_2^2 > h_3 ,$$

while

$$h_2^2 \neq h_3 ,$$

implies orthogonality in the formal sense.

Let A and B be the two roots of the quadratic

$$m^2 - h_2 m + h_2^2 - h_3 = 0, \tag{7.21}$$

therefore

$$h_n = \begin{cases} \alpha A^{n-2} + \beta B^{n-2} & \text{if } A \neq B \\ (\alpha+\beta n)A^{n-2} & \text{if } A = B , \end{cases} \qquad (n = 2,3,\ldots) \tag{7.22}$$

where α and β are arbitary function of n of period 1 .

It is clear that $A \neq 0$ and $B \neq 0$. Moreover orthogonality in the classical sense occurs if and only if A and B have the same sign. This case will be divided into two subcases according as $A = B$ or

$A \neq B$.

Case I. $A \neq B$. Equation (7.21) gives

$$h_2 = A + B, \; h_2^2 - h_3 = AB,$$

while equation (7.22) gives

$$h_2 = \alpha + \beta, \; h_3 = \alpha A + \beta B.$$

Eliminating α and β we get

$$h_n = \frac{A^n - B^n}{A-B} \qquad n = 1,2,\ldots \,. \tag{7.23}$$

Case II. $A = B$. Equation (7.21) gives

$$h_2 = 2A, \quad h_2^2 - h_3 = A^2 \,,$$

and equation (7.22) gives

$$h_2 = \alpha + 2\beta, \quad h_3 = (\alpha + 3\beta)A \,.$$

Solving we get

$$\alpha = 0 ,$$

$$\beta = A ,$$

i.e.

(7.24) $$h_n = nA^{n-1} \qquad n = 1,2,\ldots .$$

Therefore (7.24) is the limiting case of (7.23) as $A \to B$ and $H(t)$ will be given by

(7.25) $$H(t) = \frac{t}{(1-At)(1-Bt)} .$$

Now we proceed to evaluate the determinating series $A(t)$. It is clear that

(7.26) $$A(t) = \sum_{n=o}^{\infty} p_n(0)t^n.$$

We go back to (7.7'), put $x = q = 0$ and substitute for B_n and C_n by their values as given by (7.18) and (7.19) to get

$$p_{n+1}(0) = h_2 p_n(0) + (h_3 - h_2^2)p_{n-1}(0), \quad n = 0,1,2,\ldots .$$

Therefore

$$\sum_{n=o}^{\infty} p_{n+1}(0)t^{n+1} = h_2 t \sum_{n=o}^{\infty} p_n(0)t^n + (h_3-h_2^2)t^2 \sum_{n=1}^{\infty} p_{n-1}(0)t^{n-1},$$

i.e.

$$A(t) - 1 = h_2 tA(t) + (h_3-h_2^2)t^2 A(t).$$

Therefore

$$A(t) = \frac{1}{1-h_2 t+(h_2^2-h_3)t^2},$$

i.e.

$$A(t) = \frac{1}{(1-At)(1-Bt)},$$

and the case $A \to B$ is included.

Summarizing we have

<u>Theorem</u> 7.2 <u>The orthogonal (in the classical sense) polynomials</u> $\{p_n(x)\}_o^{\infty}$ <u>which are generated by</u>

(7.27) $$\sum_{n=o}^{\infty} p_n(x)t^n = A(t)\{1-xH(t)\}^{-1},$$

<u>such that</u> $A(t)$ <u>and</u> $H(t)$ <u>satisfy</u> (7.3) <u>and</u> (7.4) <u>respectively</u>, <u>are the Tchebicheff polynomials up to a linear transformation in</u> x.

For orthogonality in the formal sense it remains to consider the case when $AB < 0$. Define $\{g_n(x)\}_o^\infty$ by

$$\sum_{n=o}^{\infty} g_n(x)t^n = \frac{1}{1-2xt-t^2} .$$

Therefore if $AB < 0$ then the polynomials $\{p_n(x)\}_o^\infty$ with generating function (7.27) such that (7.3) and (7.4) are valid, are the polynomials $\{g_n(x)\}_o^\infty$ up to a linear transformation in x.

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